All-Pay Auctions with Different Forfeits

Benjamin Kang,[1] and and James Unwin[2,3]

[1] *MIT PRIMES, Massachusetts Institute of Technology, Cambridge, MA 02139*

[2] *Department of Physics, University of Illinois at Chicago, Chicago, IL 60607, USA*

[3] *Department of Physics, University of California, Berkeley & Theoretical Physics Group, LBNL & Mathematics Sciences Research Institute, Berkeley, CA 94720, USA*

In an auction each party bids a certain amount and the one which bids the highest is the winner. Interestingly, auctions can also be used as models for other real-world systems. In an all pay auction all parties must pay a forfeit for bidding. In the most commonly studied all pay auction, parties forfeit their entire bid, and this has been considered as a model for expenditure on political campaigns. Here we consider a number of alternative forfeits which might be used as models for different real-world competitions.

Keywords: All-pay Auction, Auctions, Economic Model, Competition

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1. INTRODUCTION

For thousands of years, auctions have been used as a method for selling objects and four main types of auctions have gained prominence. The first type of auction is the English auction. In this type of auction, the seller continually raises the price of the item until only one person is willing to pay, and the item is sold at this price. A second type of auction is the Dutch auction. In this auction, the seller sets an extremely high price and continually lowers it until a bidder is willing to pay. A third type of auction is the first-price sealed-bid auction. In this type, bidders all bid simultaneously and the bidder with the highest bid wins and pays that bid. A fourth type of auction is the second-price sealed-bid auction, where bidders also bid simultaneously, and the bidder with the highest bid wins but pays the second-highest bid. These four types of auctions have already been analyzed extensively

Although the four auctions previously described are the most popular ones, there are may other variations. Of particular interest are the 'war of attrition' and all-pay auction. In both of these, the bidders who do not win must pay a forfeit. In the former, the winner pays the second-highest bid, while in the latter the winner pays the highest bid. Notably, conflicts among animals [1] can be represented by the war of attrition. On the other hand all-pay auctions have be used to model the arms race [2] and war outcomes [3], also rent-seeking scenarios such as lobbying [4] or competition with sunk investments [5].

In a seminal paper Milgrom and Weber [6], building on earlier work [7–11], demonstrated mathematically the equivalence of a number of auction systems under certain assumptions and derived the expected selling prices and optimal bidding strategies. Krishna and Morgan [12] expanded on these results by calculating the bidder strategies for all-pay auctions and the war of attrition scenario. There were also multiple other papers that subsequently analyzed different variants on all-pay auctions such as the case with two bidders analyzed by Amann and Leininger [13] and the behavior of bidders in an all-pay auction with incomplete information [14, 15]. Additionally, Che and Gale [16] studied the relationship between all-pay and first-price auctions.

In the most commonly studied models of all pay auctions, parties forfeit their entire bid. An important comparison to a real world situation where both parties pay the full cost of their bids but only the winner profits is expenditure on political campaigns [17]. One can also explore variants in which the non-winning bidders pay different amounts based on all of the bids. Possibilities which we will discuss here include bidders paying a constant entrance fee, or paying a fraction of their bid. These auctions are not as prominent in commercial settings but can be used as models for many other systems, some of which we will highlight.

In this paper we first outline some relevant background in Section 2 and then proceed to extend the results of Krishna and Morgan [12] to all-pay auctions with different forfeits for the losing bidders. In Section 3 we examine auctions with an entrance fee in addition to paying the bid, both when the fee is returned to the winner and when it is not. Section 4 explores the case in which the forfeit function is a constant fraction of the original bid. For these auctions, we derive an expression for the symmetric bidding strategy in each case. Then between these types of auctions, we compare the revenue made for the seller. In Section 5 we consider the approximate behavior of bidders with an exponential forfeit as the bid grows larger. Section 6 provides a summary and a discussion of possible future research directions.

2. BACKGROUND: ALL-PAY AUCTIONS

An independent private values model is an auction in which bidders are risk-neutral and only know the value of the object to himself, with values are taken from a continuous distribution. Milgrom and Weber [6] developed such a model applicable to any symmetric auction, which we outline below.

Suppose there are n bidders all competing for a single object. Each bidder has their own information about the object, it is standard to define $X = (X_1, X_2, \ldots, X_n)$ where the components represent the information known by each bidder i, Also introduce $S = (S_1, S_2, \ldots S_m)$ which represents additional variables that affect the value of the object but are only known to the seller. Then suppose that there is a nonnegative finite function u such that $u(S, X_i, \{X_j\}_{j \neq i}) = V_i$, giving the value of the object to bidder i. The payoff for the winner is $V_i - b$ where b is the price paid.

Furthermore, denote by $f(S, X_1, \cdots, X_n)$ be the joint probability distribution of the random variables, which is symmetric in the last n variables. Note that this function f obeys the 'affiliation inequality' [18] given by $f(z \vee z')f(z \wedge z') \geq f(z)f(z')$ where $z \vee z'$ is the component-wise maximum and $z \wedge z'$ is the component-wise minimum. This implies that it is more likely for the variables to be close to each other, rather than farther apart. Then define $Y_1 = \max\{X_j\}_{j \neq 1}$ and let $f_{Y_1}(\cdot|x)$ be the conditional density of Y_1 with $x = X_1$. We denote the corresponding cumulative distribution as $F_{Y_1}(\cdot|x)$. Note the cumulative distribution of a function f at a point y is defined as the probability that the result is at most $f(y)$ and can be expressed as

$$F_{Y_1}(y|x) = \int_{-\infty}^{y} f_{Y_1}(s|x)\mathrm{d}s.$$

Moreover, the following Lemma, due to Milgrom and Weber [6], will be useful going forward

Lemma 1: $\frac{F_{Y_1}(x|z)}{f_{Y_1}(x|z)}$ is non-increasing in z.

Proof. By the affiliation inequality, for $\alpha \leq x$ and $z' \leq z$,

$$f_{Y_1}(\alpha|z)f_{Y_1}(x|z') \leq f_{Y_1}(\alpha|z')f_{Y_1}(x|z) \implies \frac{f_{Y_1}(\alpha|z)}{f_{Y_1}(x|z)} \leq \frac{f_{Y_1}(\alpha|z')}{f_{Y_1}(x|z')}.$$

Integrating both sides with respect to α from $-\infty$ to x gives

$$\frac{F_{Y_1}(x|z)}{f_{Y_1}(x|z)} \leq \frac{F_{Y_1}(x|z')}{f_{Y_1}(x|z')}.$$

□

The above model has been used to study both first-price auctions [6] and all-pay auctions [12] and will also be the focus of this work. In the classic all-pay auction in which the losers forfeit their bid, one can define a payoff function W of the following form [12]

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j \neq i} b_j \\ -b_i & b_i < \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i & b_i = \max_{j \neq i} b_j \end{cases}.$$

Below we will outline the heuristic for finding the symmetric equilibrium bidding strategy from [12]. Suppose bidders $j \neq 1$ follow the symmetric, increasing equilibrium strategy α and bidder 1 bids b with $X_1 = x$. Denote by v the expected value $E[\cdot]$ of the object to bidder 1, defined by

$$v(x, y) = E[V_1 | X_1 = x, Y_1 = y].$$

Then the expected payoff of bidder 1, denoted $\Pi(b, x)$, is given by

$$\Pi(b, x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x, y) f_{Y_1}(y|x) \mathrm{d}y - b.$$

We can obtain the maximum payoff for bidder 1 with respect to the bid by finding when the derivative with respect to b vanishes, which implies

$$v(x, \alpha^{-1}(b)) f_{Y_1}(\alpha^{-1}(b)|x) \frac{1}{\alpha'(\alpha^{-1}(b))} - 1 = 0.$$

At symmetric equilibrium, bidder 1 also follows the bidding strategy α, so $\alpha(x) = b$, which gives

$$\alpha'(x) = v(x, x) f_{Y_1}(x|x).$$

Integrating this equation one obtains that the symmetric equilibrium has the following form

$$\alpha(x) = \int_{-\infty}^{x} v(t, t) f_{Y_1}(t|t) \mathrm{d}t.$$

However, this is only a necessary condition for the bidding strategy to be a symmetric equilibrium.

In [12] Krishna & Morgan the following theorem is proved which establishes the above as the symmetric equilibrium bidding strategy for bidders in an all-pay auction:

Theorem 1. *Let $\psi(x, y) = v(x, y) f_{Y_1}(y|x)$. If $\psi(x, y)$ is increasing in x, then the formula for symmetric equilibrium function is given by*

$$\alpha(x) = \int_{-\infty}^{x} v(t, t) f_{Y_1}(t|t) \mathrm{d}t.$$

Furthermore, Milgrom & Weber [6] showed that the symmetric equilibrium bidding strategy in a first-price auction obeys a similar statement:

Theorem 2. *The function of symmetric equilibrium for a first-price auction is given by*

$$\alpha(x) = \int_{-\infty}^{x} v(s, s) \frac{f_{Y_1}(s|s)}{F_{Y_1}(s|s)} \exp\left(\int_{x}^{s} \frac{f_{Y_1}(t|t)}{F_{Y_1}(t|t)} \mathrm{d}t \right) \mathrm{d}s.$$

Additionally, from the above theorems Krishna & Morgan [12] proved that if $\psi(x, y)$ is increasing in x, then the expected revenue from an all-pay auction is at least as great as that from a first-price auction. These results are important to sellers since it can help them determine what type of auction they should use and how much they should expect to receive. In the remainder of this work we will explore how the symmetric equilibria $\alpha(x)$ of all pay auctions are impacted by changes to the forfeits that the losing bidders are required to pay by studying changes to the forfeit function W.

3. AUCTIONS WITH CONSTANT ENTRANCE FEES

In this section, we will investigate the effects of introducing a constant entrance fee to an all-pay auction, a possibility remarked in passing in [12] but not analyzed further. First, we will examine when the winner does not have their entrance fee returned, following which we consider the converse case in which the winner alone receives their entrance fee back.

Suppose that a fee c is required to enter the auction, in the case that entrance fee is not returned to the winner the expected payoff can be expressed as follows

$$W_i = \begin{cases} V_i - b_i - c & b_i > \max_{j \neq i} b_j \\ -b_i - c & b_i < \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i - c & b_i = \max_{j \neq i} b_j \end{cases}.$$

Adapting earlier arguments, we now derive a heuristic for the bidding strategy with this new form of W. Suppose bidders $j \neq 1$ follow symmetric increasing equilibrium strategy α. The expected payoff of bidder 1 making a bid b (and writing $X_1 = x$) is given by

$$\Pi(b, x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x, y) f_{Y_1}(y|x) \mathrm{d}y - b - c.$$

To understand this form of the payoff note that the integral represents the expected value of the object when bidder 1 wins and the other terms correspond to the amount bidder 1 pays, depending on whether or not they won the auction.

To proceed we maximize the payoff function, similar to previously, thus we set the derivative with respect to b to zero to obtain the condition

$$v(x, \alpha^{-1}(b)) f_{Y_1}(\alpha^{-1}(b)|x) \frac{1}{\alpha'(\alpha^{-1}(b))} - 1 = 0.$$

Interestingly, this equation does not depend on c at all, which means that the addition of a constant entrance fee to an all-pay auction does not affect the strategy if the entrance fee is paid by everyone.

Next we consider the converse case in which the entrance fee is returned to the winning bidder (as might be a model for certain gambling scenarios). In this case the expected payoff is given by

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j \neq i} b_j \\ -b_i - c & b_i < \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i & b_i = \max_{j \neq i} b_j \end{cases}.$$

Suppose bidders $j \neq 1$ follow symmetric increasing equilibrium strategy α. The expected payoff of bidder 1 making a bid b can be expressed as follows

$$\Pi(b, x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x, y) f_{Y_1}(y|x) \mathrm{d}y - b - c(1 - F_{Y_1}(\alpha^{-1}(b)|x)),$$

and thus the payoff is maximized for

$$v(x, \alpha^{-1}(b)) f_{Y_1}(\alpha^{-1}(b)|x) \frac{1}{\alpha'(\alpha^{-1}(b))} - 1 + c \cdot f_{Y_1}(\alpha^{-1}(b)|x) \frac{1}{\alpha'(\alpha^{-1}(b))} = 0.$$

Similar to the derivation of eq. (**??**), bidder 1 follows the strategy $\alpha(x)=b$, thus it follows that

$$\alpha'(x)=(v(x,x)+c)f_{Y_1}(x|x)$$

and thus

$$\alpha(x)=\int_{-\infty}^{x}(v(t,t)+c)f_{Y_1}(t|t)\mathrm{d}t=\int_{-\infty}^{x}v(t,t)f_{Y_1}(t|t)\mathrm{d}t+c\int_{-\infty}^{x}f_{Y_1}(t|t)\mathrm{d}t.$$

This implies that when the entrance fee is returned to the winner, the bidding strategy changes. This occurs because there is no longer symmetry in the forfeits. Notably, it can be observed that without this symmetry the optimal bid amount increases as the entrance fee is raised.

4. AUCTIONS WITH FRACTIONAL FORFEITS

We next analyze the effects of having the forfeit be a fraction $\beta\in(0,1)$ of each parties bid, therefore

$$W_i=\begin{cases}V_i-b_i & b_i>\mathrm{max}_{j\neq i}b_j\\ -\beta b_i & b_i<\mathrm{max}_{j\neq i}b_j\\ \frac{V_i}{\#\{k:b_k=b_i\}}-b_i & b_i=\mathrm{max}_{j\neq i}b_j\end{cases}.$$

Again suppose bidders $j\neq 1$ follow the symmetric increasing equilibrium strategy α, then the expected payoff of bidder 1 is

$$\Pi(b,x)=\int_{-\infty}^{\alpha^{-1}(b)}v(x,y)f_{Y_1}(y|x)\mathrm{d}y-bF_{Y_1}(\alpha^{-1}(b)|x)-(\beta b)(1-F_{Y_1}(\alpha^{-1}(b)|x)). \tag{1}$$

It follows that the bid b that maximizes eq. (1) satisfies the following condition

$$v(x,\alpha^{-1}(b))f_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))}-F_{Y_1}(\alpha^{-1}(b)|x)-bf_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))}$$
$$-\beta(1-F_{Y_1}(\alpha^{-1}(b)|x))+(\beta b)\cdot f_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))}=0.$$

Multiplying both sides of the above equation by $\alpha'(\alpha^{-1}(b))$ then taking $\alpha(x)=b$ gives

$$\beta\alpha'(x)+(1-\beta)\alpha'(x)F_{Y_1}(x|x)+(1-\beta)\alpha(x)f_{Y_1}(x|x)=v(x,x)f_{Y_1}(x|x). \tag{2}$$

Observe that the above is a first order differential equation in $\alpha(x)$. Solving this equation, we obtain

$$\alpha(x)=\int_{-\infty}^{x}v(s,s)\frac{\mathrm{d}L(s,x)}{1-\beta},$$

where

$$L(s,x)=\exp\left((1-\beta)\int_{x}^{s}\frac{f_{Y_1}(t|t)}{\beta+(1-\beta)F_{Y_1}(t|t)}\mathrm{d}t\right).$$

Moreover, this can then be rewritten as follows

$$\alpha(x)=\int_{-\infty}^{x}v(s,s)\frac{f_{Y_1}(s|s)}{\beta+(1-\beta)F_{Y_1}(s|s)}\exp\left(-(1-\beta)\int_{s}^{x}\frac{f_{Y_1}(t|t)}{\beta+(1-\beta)F_{Y_1}(t|t)}\mathrm{d}t\right)\mathrm{d}s.$$

It is interesting that for $\beta=0$ we obtain the strategy for first-price auctions [6] and taking $\beta=1$ yields the strategy for the classic all-price auction in [12].

Theorem 3. *When $\alpha(x)$ is as defined above, it is a symmetric equilibrium.*

Proof. Let $t(x) = v(x, x)$, then by integration by parts α can be rewritten as

$$\alpha(x) = \frac{v(x, x) - \int_{-\infty}^{x} L(s, x)\mathrm{d}t(s)}{1 - \beta}. \tag{3}$$

We adapt an argument in [6][1] in which it was observed that $L(\alpha|x)$ is decreasing in x and $v(x, x)$ is increasing, so $\alpha(x)$ will grow with increasing x. First suppose $\alpha(x)$ is continuous, then we can assume $\alpha(x)$ is differentiable without loss of generality by monotonically rescaling bidder estimates. To show that $\alpha(x)$ is the optimal bid, we find the maximum of the payoff function

$$\frac{\partial}{\partial b}\Pi(\alpha(z), x) = \frac{f_{Y_1}(z|x)}{\alpha'(z)}\left(v(x, z) - (1 - \beta)\alpha(z) - \alpha'(z)\frac{(1 - \beta)F_{Y_1}(z|x)}{f_{Y_1}(z|x)}\right) - \beta.$$

Applying Lemma 1 and the fact that $v(x, z)$ is increasing it follows that $\frac{\partial}{\partial b}\Pi(\alpha(z), x)$ has the same sign as $z - x$, this implies that $\Pi(\alpha(z), x)$ is maximized for $z = x$.

It remains to check cases in which α is discontinuous at some point x. In this case for any positive ϵ, the following expression is infinite

$$\int_{x}^{x+\epsilon} \frac{(1 - \beta)f_{Y_1}(s|s)}{\beta + (1 - \beta)F_{Y_1}(s|s)} \sim \infty.$$

It then follows that

$$\begin{aligned}
\int_{x}^{x+\epsilon} \frac{(1 - \beta)f_{Y_1}(s|s)}{\beta + (1 - \beta)F_{Y_1}(s|s)} &\leq \int_{x}^{x+\epsilon} \frac{f_{Y_1}(s|s)}{\beta F_{Y_1}(s|s) + (1 - \beta)F_{Y_1}(s|s)} \\
&= \int_{x}^{x+\epsilon} \frac{f_{Y_1}(s|s)}{F_{Y_1}(s|s)} \\
&\leq \int_{x}^{x+\epsilon} \frac{f_{Y_1}(s|x + \epsilon)}{F_{Y_1}(s|x + \epsilon)} \\
&= \ln(F_{Y_1}(x + \epsilon|x + \epsilon)) - \ln(F_{Y_1}(x|x + \epsilon)).
\end{aligned}$$

For the last expression to be infinite, it is required that $F_{Y_1}(x|x+\epsilon) = 0$, which is a statement proved in Theorem 14 of [6]. Therefore, $\alpha(x)$ as given in eq. (3) is an equilibrium for this auction. □

Theorem 4. *The expected revenue generated for the seller of an all-pay auction with fractional cost is always less than when $\beta = 1$ if $f(y|x)$ is increasing in x.*

Proof. Let $\alpha_\beta(x)$ be the equilibrium bid for a specific value of β. Notice that the expected payment of a bidder is

$$\begin{aligned}
e_\beta(x) &= (F_{Y_1}(x|x) + \beta(1 - F_{Y_1}(x|x)))\alpha_\beta(x) \\
&= \int_{-\infty}^{x} v(s, s)f_{Y_1}(s|s)\frac{\beta + (1 - \beta)F_{Y_1}(x|x)}{\beta + (1 - \beta)F_{Y_1}(s|s)} \exp\left(-\int_{x}^{s} \frac{(1 - \beta)f_{Y_1}(t|t)}{\beta + (1 - \beta)F_{Y_1}(t|t)}\mathrm{d}t\right) \mathrm{d}s.
\end{aligned}$$

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[1] Specifically, Theorem 14 of [6] which studied a case corresponding to the $\beta = 0$ case of our generalized set-up.

Since $f_{Y_1}(y|x)$ is increasing in x, it follows that $\beta/f_{Y_1}(y|x)$ is decreasing in x. Combined with Lemma 1, this implies that $\frac{f_{Y_1}(y|x)}{\beta+(1-\beta)F_{Y_1}(y|x)}$ is increasing in x and therefore

$$\begin{aligned} -\int_s^x \frac{(1-\beta)f_{Y_1}(t|t)}{\beta+(1-\beta)F_{Y_1}(t|t)}\mathrm{d}t &\leq -\int_s^x \frac{(1-\beta)f_{Y_1}(t|s)}{\beta+(1-\beta)F_{Y_1}(t|s)}\mathrm{d}t \\ &= \ln(\beta+(1-\beta)F_{Y_1}(s|s)) - \ln(\beta+(1-\beta)F_{Y_1}(x|s)) \\ &\leq \ln(\beta+(1-\beta)F_{Y_1}(s|s)) - \ln(\beta+(1-\beta)F_{Y_1}(x|x)) \ , \end{aligned}$$

where the last inequality comes from the fact that $F_{Y_1}(y|x)$ is non-increasing in x. It follows that

$$\begin{aligned} e_\beta(x) &\leq \int_{-\infty}^x v(s,s)f_{Y_1}(s|s)\frac{\beta+(1-\beta)F_{Y_1}(x|x)}{\beta+(1-\beta)F_{Y_1}(s|s)}\exp\left(\ln\left(\frac{\beta+(1-\beta)F_{Y_1}(s|s)}{\beta+(1-\beta)F_{Y_1}(x|x)}\right)\right) \\ &\leq \int_{-\infty}^x v(s,s)f_{Y_1}(s|s) = e_1(x). \end{aligned}$$

□

Notably, this shows that the expected amount paid by a bidder in an auction where $\beta \leq 1$ is at most the expected price paid by a bidder in the original all-pay auction. Since this holds for each bidder, it follows for the expected revenue earned by the seller as well.

To summarize, in this section we have proved the equilibrium bidding strategy for the all-pay auction with fractional forfeit. We also showed that each of these auctions does not generate as much revenue as the all-pay auction with complete bid forfeit. However, an ordering between two auctions with different values of β is yet to be determined.

5. AUCTIONS WITH EXPONENTIAL FORFEITS

We next consider the interesting case in which the losers must pay an exponentially large forfeit, as described by the following expected payoff function

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j\neq i} b_j \\ -e^{b_i} & b_i < \max_{j\neq i} b_j \\ \frac{V_i}{\#\{k: b_k = b_i\}} - b_i & b_i = \max_{j\neq i} b_j \end{cases} .$$

Suppose bidders $j \neq 1$ follow symmetric increasing equilibrium strategy α, then the expected payoff of bidder 1 with bid b is

$$\Pi(b,x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x,y)f_{Y_1}(y|x)\mathrm{d}y - bF_{Y_1}(\alpha^{-1}(b)|x) - e^b(1-F_{Y_1}(\alpha^{-1}(b)|x)).$$

The bid that maximizes the payoff, is given by the critical point with respect to b, given by

$$\begin{aligned} v(x,\alpha^{-1}(b))f_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))} - F_{Y_1}(\alpha^{-1}(b)|x) - bf_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))} \\ - e^b(1-F_{Y_1}(\alpha^{-1}(b)|x)) + e^b \cdot f_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))} = 0. \end{aligned}$$

Multiplying both sides of the above equation by $\alpha'(\alpha^{-1}(b))$ and evaluating for $\alpha(x) = b$ gives

$$e^{\alpha(x)}\alpha'(x) + (1 - e^{\alpha(x)})\alpha'(x)F_{Y_1}(x|x) + (\alpha(x) - e^{\alpha(x)})f_{Y_1}(x|x) = v(x,x)f_{Y_1}(x|x),$$

which we rearrange to find an expression for α'

$$\alpha'(x) = \frac{(v(x,x) + e^{\alpha(x)} - \alpha(x))f_{Y_1}(x|x)}{e^{\alpha(x)} + (1 - e^{\alpha(x)})F_{Y_1}(x|x)}.$$

The differential equation which arises cannot be solved analytically. However, for large bids one may consider the approximate behavior for in the large b limit, which implies that

$$\alpha'(x) \approx \frac{f_{Y_1}(x|x)}{1 - F_{Y_1}(x|x)}.$$

It follows that the equilibrium strategy in this limit is given by

$$\alpha(x) \approx \int_{-\infty}^{x} \mathrm{d}t \left(\frac{f_{Y_1}(t|t)}{1 - F_{Y_1}(t|t)} \right).$$

Interestingly, observe that this is independent of the function v, which represents the expected value of the object to bidder 1.

As an explicit example, consider the case of two bidders where we denote bidder 1's signal by x and bidder 2's signal by y, and suppose

$$f(x,y) = \frac{4}{5}(1 + xy),$$

where f is defined on $[0,1] \times [0,1]$. This implies that $f_{Y_1}(y|x) = \frac{2+2xy}{2+x}$ and $F_{Y_1}(y|x) = \frac{2y+xy^2}{2+x}$, thus

$$\alpha(x) = \int_0^x \left(\frac{2 + 2t^2}{2 - t - t^3} \right) \mathrm{d}t = \int_0^x \left(\frac{1}{1-t} - \frac{t}{2 + t + t^2} \right) \mathrm{d}t.$$

This function behaves very similarly to $-\ln(1-x)$ since the second term in the integral is negligible. Notice that this function increases slowly at first but then begins to grows increasingly rapidly. This is indicative of the optimal strategy for successful parties in auctions with exponential forfeits, specifically, that likely winners are those that bid significantly more than the typical bidder.

6. SUMMARY

In this work we have investigated the impact of changing the forfeit function in all pay auctions. We highlighted that the addition of a constant entrance fee does not affect the bidding strategy unless the fee is returned to the winner. When the forfeit is instead a fraction of the bid, we showed that the revenue generated by the seller is increasing with the fraction. Lastly, when the forfeit is exponential, the bidding strategy quickly approaches infinity and it was argued that successful bidders will be those that bid significantly more than the typical bid.

Perhaps one of the more relevant applications of all pay auctions to current events is that it provides a potential model of trade wars. A trade war occurs when two countries create tariffs in response to the other country. Each country is looking to limit their opponent economically, but it also harms their consumers, as the price of certain items will increase due to lack of imported materials. Surprisingly, auction models of trade wars appear to be absent in the literature (although there are some game theory studies, e.g. [19]), however an all pay auction seems an ideal model to study such international competition. We will return to outline an all-pay model of trade wars in a dedicated paper. Indeed, this is very relevant to our present day, with the US and China, the two largest economies in the world, currently locked in a trade war.

Moreover, it would also be interesting to consider all-pay auctions with a wider range of forfeit functions, such as logarithmic, polynomial, or constant functions. Likely this is most readily implemented by considering fractional forfeits for all parties with the forfeit value differing depending on the ranking of each parties bid, with the forfeits following a specified distribution. One might also explore the difference in results if the bidders are risk-averse rather than risk-neutral, or the effects of multiple prizes on the results, generalizing [20] to different forfeit functions. These forms of auctions all exist in the real world, thus it is important to work towards fully understanding them.

Acknowledgements. This research was undertaken as part of the MIT-PRIMES program. JU is grateful for support from the National Science Foundation grant DMS-1440140 while in residence at the Mathematics Sciences Research Institute, Berkeley during Fall 2019.

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