A Study of the Carathéodory Conjecture through the Perturbation of a Rotationally Symmetric Surface

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Abstract

Carathéodory's well-known conjecture states that every sufficiently smooth, closed convex surface in three dimensional Euclidean space admits at least two umbilic points. It has been established that the conjecture is true for all rotationally symmetric surfaces; in this paper, we investigate the umbilic points of a family of surfaces without rotational symmetry, and compute their indices. In particular, we find that the family of surfaces of the form $ax^{2k} + by^{2k} + cz^{2k} = 1$ with $a, b, c > 0$, $k \in \mathbb{Z}_{>1}$ admit 14 umbilic points: six of one known form and eight of another. For many tested values of a, b, c, k, such umbilic points have indices $-\frac{1}{2}$ and 1, respectively. We also explore the dependence of the umbilic points on the parameter ϵ of the surface $\frac{x^2}{4} + \epsilon x^4 + \frac{y^2}{4} + \epsilon y^4 + \frac{z^2}{9} = 1$.

Keywords: Umbilical point, Carathéodory conjecture, convex surface, index, principal field lines

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Introduction

The Carathéodory conjecture is a famous conjecture posed by Constantin Carathéodory in a 1924 session of the Berlin mathematical society. It has subsequently appeared in many works and problem lists since then, one notable example of which is the problem list of S.T. Yau [1] (page 684, problem 64). Before we state the conjecture, we briefly recall several concepts.

Consider a point p on a closed convex smooth surface M in $\mathbf{R}^3$ (in this paper, we will use "smooth" to denote C^∞). Every plane containing the unit normal vector at p defines a normal curvature, the maximum and minimum of which are denoted the principal curvatures k_1, k_2. The directions at which k_1, k_2 point are denoted the principal directions, and are always orthogonal. At an umbilic point x, the principal curvatures are equal, i.e., $k_1 = k_2$. In other words, M is locally spherical at x, and it follows that x is a singularity of the principal direction field. The gaussian and mean curvatures are defined as $k_1 k_2$ and $\frac{k_1+k_2}{2}$, respectively. In this paper, we use E, F, G to denote the coefficients of the first fundamental form I and e, f, g to denote the coefficients of the second fundamental form II. The principal curvatures and directions at each point p are given by the shape operator S, defined in terms of the fundamental form coefficients:

$$S = (EG - F^2)^{-1} \begin{pmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{pmatrix}$$

The index of an umbilic point x is defined to be the index of its principal direction field about x. See [2] (15.1, page 153) for a formal definition.

The Poincaré-Hopf index theorem states that the sum of the indices of all umbilic points on a surface equals its Euler characteristic. All closed convex sufficiently smooth surfaces in $\mathbf{R}^3$ have Euler characteristic 2. Carathéodory's conjecture is closely tied to Loewner's conjecture, which states that every isolated umbilic point has index less than or equal to 1. A proof of Loewner's conjecture, together with the Poincaré-Hopf Theorem, implies the truth of Carathéodory's conjecture; most attempts at proving Carathéodory's conjecture take this route. Bol [3] and Hamburger [4, 5, 6] were the first to prove the conjecture for the real analytic case, although doubts were later expressed and the results were reexamined by Klotz [7]. Ghomi and Howard have written a paper in which they use Mobius inversions to create closed convex smooth and umbilic free surfaces in the complement of one point (and get arbitrarily close to a sphere). Further historical results can be found in [8].

The Carathéodory conjecture is nearly one century old, and has resisted numerous attacks even for the real analytic case. To settle the conjecture, one really needs to understand some nontrivial examples. The first type of surfaces are perhaps the simplest non-rotationally symmetric smooth convex surfaces. The second type surfaces are small perturbations of ellipsoids. These two types of surfaces are all nontrivial examples. It is therefore very natural for us to study them.

It is known that all rotationally symmetric surfaces have at least two umbilic points. See, for example, Hilbert and Cohn-Vossen [9] (page 203). In this paper, we aim to shed light on the conjecture by exploring the umbilic points of several non-rotationally symmetric surfaces. Umehara and Yamada [2] (page 163, Example 15.8) discuss the example of the non-rotationally symmetric ellipsoid $ax^2 + by^2 + cz^2 = 1$ with a, b, c distinct. This example has 4 umbilic points with index $\frac{1}{2}$. In our paper, we generalize this example to surfaces of the form $ax^{2k} + by^{2k} + cz^{2k} = 1$. We compute the number and location of the umbilic points of such surfaces, as well as their indices for several tested values of a, b, c, k. We also explore umbilic points of the surface $\frac{x^2}{4} + \epsilon x^4 + \frac{y^2}{4} + \epsilon y^4 + \frac{z^2}{9} = 1$ and show that the location and index of the umbilic points do not depend on ϵ.

A Simple Non-Rotationally Symmetric Surface

In this section, we explore the family of surfaces of the form $ax^{2k} + by^{2k} + cz^{2k} = 1$, where $a, b, c > 0, k \in \mathbb{Z}_{>1}$. We compute the number of umbilic points of such surfaces, as well as their locations and indices.

Theorem 1.1. All surfaces of the form $ax^{2k} + by^{2k} + cz^{2k} = 1$, where $a, b, c > 0, k \in \mathbb{Z}_{>1}$ admit 14 umbilic points: 6 of the form $\left\{(\pm(\frac{1}{a})^{\frac{1}{2k}}, 0, 0), (0, \pm(\frac{1}{b})^{\frac{1}{2k}}, 0), (0, 0, \pm(\frac{1}{c})^{\frac{1}{2k}})\right\}$ and 8 of the form $\left\{ \left(\pm\left(\frac{bc}{a}\right)^{\frac{1}{2k}} \left(\frac{1}{bc+ca+ab}\right)^{\frac{1}{2k}}, \pm\left(\frac{ab}{c}\right)^{\frac{1}{2k}} \left(\frac{1}{bc+ca+ab}\right)^{\frac{1}{2k}}, \pm\left(\frac{ac}{b}\right)^{\frac{1}{2k}} \left(\frac{1}{bc+ca+ab}\right)^{\frac{1}{2k}}\right) \right\}$.

For reference, we include two instances of the surface below. Visible umbilic points are highlighted in red.

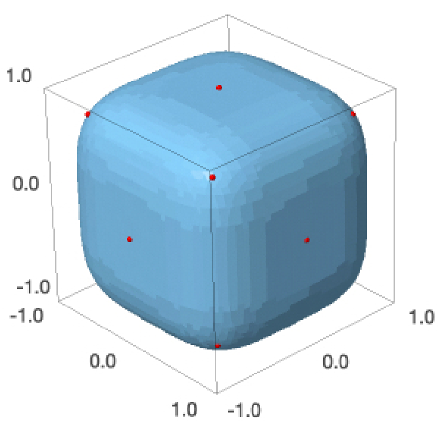


Figure 1: $x^4 + y^4 + z^4 = 1$

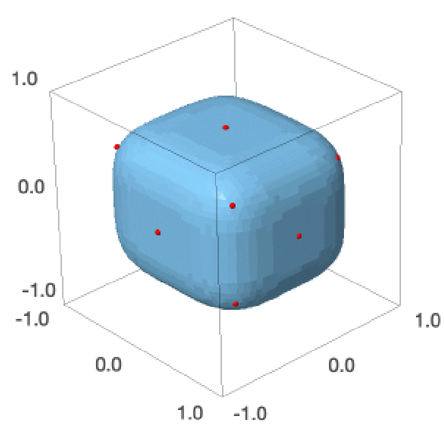


Figure 2: $2x^4 + 3y^4 + 5z^4 = 1$

Proof. We split the proof into two steps. In the first, we prove that the surface satisfies the given requirements. In the second, we prove the number and location of the umbilic points. In an additional third step, we illustrate the indices of the umbilic points for several instances of the surface and conjecture as to its general form.

Step 1. First, we prove all such surfaces are closed, convex, and smooth. To do so, we first prove all such surfaces are regular. It is possible to express all such surfaces as $f^{-1}(0)$, where

$$f(x, y, z) = ax^{2k} + by^{2k} + cz^{2k} - 1.$$

Note that f is infinitely differentiable. Also, 0 is a regular value of f, since its partial derivatives $f_x = 2akx^{2k-1}$, $f_y = 2bky^{2k-1}$, $f_z = 2ckz^{2k-1}$ only vanish simultaneously at $(0, 0, 0)$, which is not contained in $f^{-1}(0)$. It follows from the inverse function theorem [10] that all such surfaces are regular and smooth.

In order to check for convexity, we first compute the first and second fundamental coefficients. We parameterize the surface as $\mathcal{S} = (u, v, (1-au^{2k}-bv^{2k})^{\frac{1}{2k}})$. Refer to appendix [1] for detailed calculations of the fundamental form coefficients. They are given below.

$$E = \frac{a^2u^{4k-2}}{c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}}}+1, \quad F = \frac{abu^{2k-1}v^{2k-1}}{c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}}}, \quad G = \frac{b^2v^{4k-2}}{c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}}}+1$$

$$e = \frac{\frac{au^{2k-2}(2k-1)(-bv^{2k}+1)}{1-au^{2k}-bv^{2k}}}{\sqrt{c^{\frac{1}{k}}(1-au^{2k}-bv^{2k})^{2-\frac{1}{k}}+a^2u^{4k-2}+b^2v^{4k-2}}}, \quad f = \frac{\frac{ab(2k-1)u^{2k-1}v^{2k-1}}{1-au^{2k}-bv^{2k}}}{\sqrt{c^{\frac{1}{k}}(1-au^{2k}-bv^{2k})^{2-\frac{1}{k}}+a^2u^{4k-2}+b^2v^{4k-2}}},$$

$$g = \frac{\frac{bv^{2k-2}(2k-1)(-au^{2k}+1)}{1-au^{2k}-bv^{2k}}}{\sqrt{c^{\frac{1}{k}}(1-au^{2k}-bv^{2k})^{2-\frac{1}{k}}+a^2u^{4k-2}+b^2v^{4k-2}}}$$

These coefficients exist as long as $1-au^{2k}-bv^{2k} \neq 0$ and $a^2u^{4k-2}+b^2v^{4k-2}+c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}} \neq 0$. The first equation is the boundary of our parameterization: we do not need to worry about this since it is possible to switch a, b, c to check for umbilic points along this boundary. The second equation has no solutions (since the left side is the sum of square roots that cannot simultaneously be 0).

Now, to prove all such surfaces are convex, we introduce a lemma.

Lemma 1.1 (Convexity)**.** A closed surface in $\mathbf{R}^3$ is convex if and only if its Gaussian curvature is nonnegative everywhere.

Proof. The proof of this lemma is due to the Chern-Lashof Theorem; see [10] (p. 387, Remark 3). ☐

We compute the Gaussian curvature K. By symmetry, we only need to consider the open parameterization, so that $au^{2k}+bv^{2k}<1$.

$$K = \frac{\det(II)}{\det(I)} = \frac{eg-f^2}{EG-F^2}$$

Note that since $1-au^{2k}-bv^{2k}>0$ and $\sqrt{c^{\frac{1}{k}}(1-au^{2k}-bv^{2k})^{2-\frac{1}{k}}+a^2u^{4k-2}+b^2v^{4k-2}}>0$,

$$\begin{aligned}
\operatorname{sgn}(eg-f^2) &= \operatorname{sgn}(abu^{2k-2}v^{2k-2}(2k-1)^2(1-au^{2k})(1-bv^{2k}) - a^2b^2(2k-1)^2u^{4k-2}v^{4k-2}) \\
&= \operatorname{sgn}(abu^{2k-2}v^{2k-2}(2k-1)^2((1-au^{2k})(1-bv^{2k}) - abu^{2k}v^{2k})) \\
&= \operatorname{sgn}(abu^{2k-2}v^{2k-2}(2k-1)^2(1-au^{2k}-bv^{2k})) \\
&\geq 0.
\end{aligned}$$

Similarly, since $c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}} \geq 0$,

$$\operatorname{sgn}(EG-F^2) = \operatorname{sgn}(c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}}(a^2u^{4k-2}+b^2v^{4k-2}+c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}})) = 1.$$

So, $\operatorname{sgn}(K) \geq 0$ and all such surfaces are convex.

Step 2. We now compute the number of umbilic points of all such surfaces. To do so, we first introduce a lemma.

Lemma 1.2 (Weingarten matrix)**.** The umbilic points of a surface occur precisely where the Weingarten matrix

$$C = A^{-1}B = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

is a scalar multiple of the identity matrix.

Proof. This is well-known; see [2] (Proposition 9.6, page 94). □

We then solve the following equations manually:

$$C[0][1] = C[1][0] = 0 \tag{1}$$

$$C[0][0] = C[1][1] \tag{2}$$

To verify that C exists, we check that $EG - F^2 \neq 0$. This quantity is equal to 0 only when

$$(a^2u^{4k-2} + c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{2-\frac{1}{k}})(b^2v^{4k-2} + c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{2-\frac{1}{k}}) - (abu^{2k-1}v^{2k-1})^2 = 0 \Rightarrow$$
$$c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{2-\frac{1}{k}}(a^2u^{4k-2} + b^2v^{4k-2} + c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{2-\frac{1}{k}}) = 0 \Rightarrow$$
$$1 - au^{2k} - bv^{2k} = 0 \quad \text{or} \quad a^2u^{4k-2} + b^2v^{4k-2} + c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{2-\frac{1}{k}} = 0.$$

Again, we do not need to worry about these two equations. Now, we expand and simplify C to get

$$C[0][0] = \frac{Ge - Ff}{EG - F^2}, \quad C[0][1] = \frac{Gf - Fg}{EG - F^2}, \quad C[1][0] = \frac{Ef - eF}{EG - F^2}, \quad C[1][1] = \frac{Eg - Ff}{EG - F^2}.$$

Equation (1) becomes $Gf = Fg$ and $Ef = eF$. We rearrange and simply to get

$$Gf = Fg \Rightarrow abu^{2k-1}v^{2k-1}(b^2v^{4k-2} + c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{2-\frac{1}{k}} - bv^{2k-2}(-au^{2k} + 1)) = 0 \Rightarrow$$
$$abu^{2k-1}v^{2k-1}(c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{2-\frac{1}{k}} - bv^{2k-2}(-au^{2k} - bv^{2k} + 1)) = 0 \Rightarrow$$
$$u^{2k-1}v^{2k-1}(-au^{2k} - bv^{2k} + 1)(c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{1-\frac{1}{k}} - bv^{2k-2}) = 0.$$

Similarly, $Ef = eF$ becomes

$$u^{2k-1}v^{2k-1}(-au^{2k} - bv^{2k} + 1)(c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{1-\frac{1}{k}} - au^{2k-2}) = 0$$

Equation (2) becomes $Ge = Eg$. We get

$$(b^2v^{4k-2} + c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{2-\frac{1}{k}})au^{2k-2}(-bv^{2k} + 1)$$
$$= (a^2u^{4k-2} + c^{\frac{1}{k}}(-au^{2k} - bv^{2k} + 1)^{2-\frac{1}{k}})bv^{2k-2}(-au^{2k} + 1)$$

Simultaneously solving these equations gives the solutions

$$(u, v) = \quad (0, 0), \quad \frac{(bc)^{\frac{1}{k-1}}}{(ab)^{\frac{1}{k-1}} + (bc)^{\frac{1}{k-1}} + (ca)^{\frac{1}{k-1}}} \left(\pm\left(\tfrac{1}{a}\right)^{\frac{1}{2k}}, \pm\left(\tfrac{1}{b}\right)^{\frac{1}{2k}}\right).$$

Taking advantage of symmetry gives the following general form for the umbilic points:

$$(x,y,z) = \Big(\pm\left(\tfrac{1}{a}\right)^{\frac{1}{2k}},0,0\Big),\quad \Big(0,\pm\left(\tfrac{1}{b}\right)^{\frac{1}{2k}},0\Big),\quad \Big(0,0,\pm\left(\tfrac{1}{c}\right)^{\frac{1}{2k}}\Big),$$
$$\frac{(bc)^{\frac{1}{k-1}}}{(ab)^{\frac{1}{k-1}}+(bc)^{\frac{1}{k-1}}+(ca)^{\frac{1}{k-1}}}\left(\pm\left(\tfrac{1}{a}\right)^{\frac{1}{2k}},\pm\left(\tfrac{1}{b}\right)^{\frac{1}{2k}},\pm\left(\tfrac{1}{c}\right)^{\frac{1}{2k}}\right)$$

□

Step 3. To find the indices of the umbilic points, we study the shape of the principal direction field near singularities.

Lemma 1.3 (Lines of curvature)**.** The equation for lines of curvature can be written as

$$\begin{vmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0.$$

Expanding gives an alternate form:

$$(fE-eF)(u')^2 + (gE-eG)u'v' + (gF-fG)(v')^2 = 0.$$

Proof. See [10] (section 3-3, page 161). □

We compute the quantities $fE-eF$, $gE-eG$, $gF-fG$. Dividing each by $2k-1$ gives

$$fE-eF = (a^2u^{4k-2}+c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}})abu^{2k-1}v^{2k-1} - a^2bu^{4k-3}v^{2k-1}(-bv^{2k}+1),$$
$$gE-eG = bv^{2k-2}(-au^{2k}+1)(a^2u^{4k-2}+c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}})$$
$$- au^{2k-2}(-bv^{2k}+1)(b^2v^{4k-2}+c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}}),$$
$$gF-fG = (-au^{2k}+1)(ab^2u^{2k-1}v^{4k-3}) - abu^{2k-1}v^{2k-1}(b^2v^{4k-2}+c^{\frac{1}{k}}(-au^{2k}-bv^{2k}+1)^{2-\frac{1}{k}}).$$

We now divide both sides by $(v')^2$. Viewing u as a function of v, we plug the differential equation into the computer algebra software Mathematica using its numerical differential equation solving method NDSolve. In the following example, we set $a=b=c=1, k=2$ and initial conditions $(v,u)=(0,0.7)$ and $v\in[-0.699999, 0.699999]$ to avoid singularities at $v=-0.7, 0.7$. We allow for 20 digits of precision and set "SolveDelayed" to "True" to avoid singularities.

In[1]:

```
a=1; b=1; c=1; k=2

(* X = fE-eF *)
X = (a^2*(u[v])^(4k-2)+c^(1/k)(-a*(u[v])^(2k)-b*v^(2k)+1)^(2-(1/k)))a*b*(u[v])^(2k-1)v
    ^(2k-1)-a^2*b*(u[v])^(4k-3)v^(2k-1)(-b*v^(2k)+1)

(* Y = gE-eG *)
Y = b*v^(2k-2)(-a*(u[v])^(2k)+1)(a^2*(u[v])^(4k-2)+ c^(1/k)(-a*(u[v])^(2k)-b*v^(2k)+1)
    ^(2-(1/k)))- a*(u[v])^(2k-2)(-b*v^(2k)+1)(b^2*v^(4k-2)+c^(1/k) (-a*(u[v])^(2k)-b*v
    ^(2k)+1)^(2-(1/k)))
```

```
(* Z = gF-fG*)
Z = (-a*(u[v])^(2 k) + 1) (a*b^2*(u[v])^(2 k - 1) v^(4 k - 3)) - a*b*(u[v])^(2 k - 1)*
    v^(2 k - 1) (b^2*v^(4 k - 2) + c^(1/k) (-a*(u[v])^(2 k) - b*v^(2 k) + 1)^(2 - (1/k
    )))

(* Solve the DE with machine precision *)
lowsol = NDSolve[{X(u'[v])^2 + Y(u'[v]) + Z == 0, u[0] == 0.7}, u, {v, -0.699999,
    0.699999}, SolveDelayed -> True, InterpolationOrder -> All]

(* Solve with higher precision *)
highsol = NDSolve[{X(u'[v])^2 + Y(u'[v]) + Z == 0, u[0] == 0.7}, u, {v, -0.699999,
    0.699999}, SolveDelayed -> True, InterpolationOrder -> All, PrecisionGoal -> 20]

(* Plot results *)
Plot[Evaluate[u[v] /. %], {v, -0.699999, 0.699999}, PlotRange -> {{-1, 1}, {-1, 1}}]
```

Out[1]:

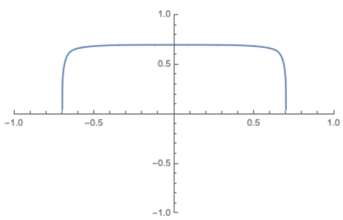


Figure 3:
$(v, u) = (0, 0.7)$
$\{v, -0.7, 0.7\}$

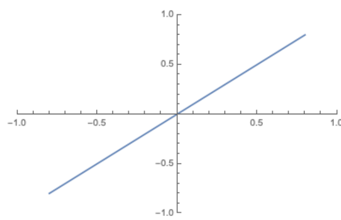


Figure 4:
$(v, u) = (0.8, 0.8)$
$\{v, -0.8, 0.8\}$

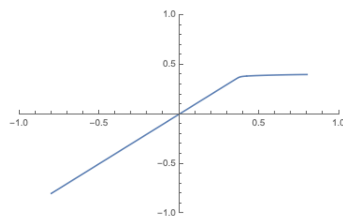


Figure 5:
$(v, u) = (0.8, 0.4)$
$\{v, -0.8, 0.8\}$

Overlaying these graphs with other solution curves gives an image of the lines of curvature near $(0, 0, 1)$ on the surface $x^4 + y^4 + z^4 = 1$. We can test a variety of surfaces by changing the input values for a, b, c, k: refer to appendix [4] for results. In particular, we see that setting $a = b = 1, k = 2$ and testing values of c up to 100 gives the same approximate principal field lines. In all of the demonstrated cases, the shape of the lines of curvature remain the same: thus, the indices of the umbilic points remain the same, and are independent of a, b, c, or k. To this end, we offer the following proposition.

Proposition 1.1. The index of the umbilic points on the surface $ax^{2k} + by^{2k} + cz^{2k} = 1$ for $a, b, c > 0, k \in \mathbb{Z}_{>1}$ are independent of a, b, c, and k.

At the umbilic points, the vector fields look approximately as follows:

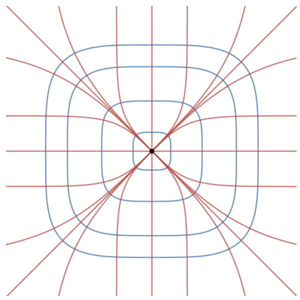


Figure 6: Vector field with index 1.

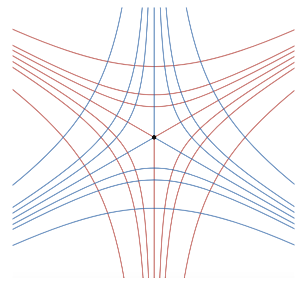


Figure 7: Vector field with index -$\frac{1}{2}$.

We can deduce by inspection that these vector fields have index 1 and $-\frac{1}{2}$, respectively. For verification, we compare Figure 7 with [2] (section 15, page 156) and check that these values agree with the Poincaré-Hopf index theorem.

Remark 1.1. This theorem readily extends to surfaces of the form $ax^{2k} + by^{2k} + cz^{2k} = R$ for $R > 0$ through a scaling of the constants a, b, c.

Remark 1.2. Note that surfaces of the form $ax^{2k+1} + by^{2k+1} + cz^{2k+1} = 1$ are not closed, since any of x, y, z can extend infinitely in the negative direction. Thus, such surfaces are not of interest, and it is safe to claim we have investigated the umbilic points of all surfaces of the form $ax^k + by^k + cz^k = 1$ with $a, b, c > 0, k \in \mathbb{Z}^+$.

Verification

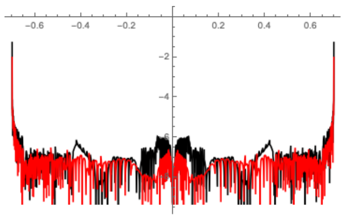
We can verify the results of NDSolve by plotting the logs of the residuals. Note that solving with higher precision (red) improves upon error, as compared to machine precision (black).

In[2]:

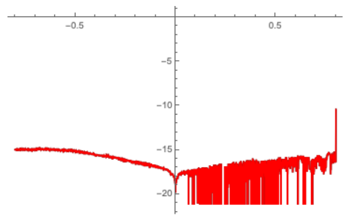
```
(*Define residual function*)
residual[v_] = X(u'[v])^2 + Y(u'[v]) + Z

(*Plot log of residuals*)
Plot[Evaluate[RealExponent[{residual[v] /. lowsol, residual[v] /. highsol}]], {v,
    -0.699999, 0.699999}, PlotStyle -> {GrayLevel[0], RGBColor[1, 0, 0]}, AxesOrigin
    -> {0, 0}]
```

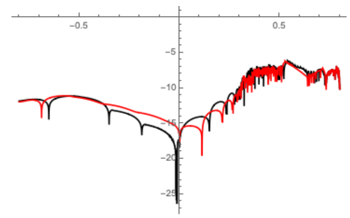
Out[2]:



$(v, u) = (0, 0.7)$
$\{v, -0.7, 0.7\}$



$(v, u) = (0.8, 0.8)$
$\{v, -0.8, 0.8\}$



$(v, u) = (0.8, 0.4)$
$\{v, -0.8, 0.8\}$

The above graphs correspond to the outputs 3, 4, 5 respectively. We see that the error mostly remains smaller than a 10^{-5} magnitude.

A Perturbation of the Ellipsoid

In this section, we explore surfaces of the form $\frac{x^2}{4} + \epsilon x^4 + \frac{y^2}{4} + \epsilon y^4 + \frac{z^2}{9} = 1$. We investigate the dependence of the number and location of umbilic points on ϵ. Again, we include two instances of the surface below, with visible umbilic points highlighted in red.

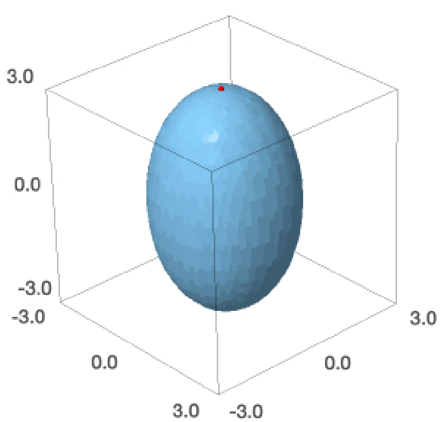


Figure 8: $\epsilon = 0.01$

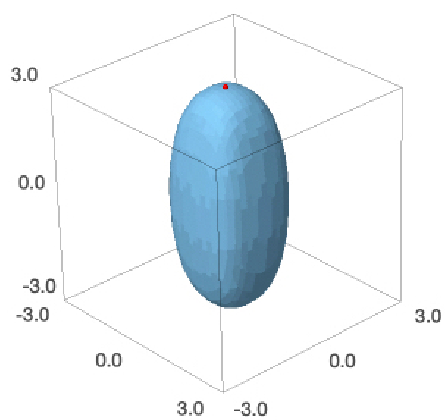


Figure 9: $\epsilon = 0.2$

Note that when $\epsilon = 0$, this surface reduces to the ellipsoid of revolution $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1$.

Theorem 3.1. The surface $\frac{x^2}{4} + \epsilon x^4 + \frac{y^2}{4} + \epsilon y^4 + \frac{z^2}{9} = 1$ where $\epsilon \geq 0$ admits 2 umbilic points of the form $(0, 0, \pm 3)$ each with index 1.

Proof. Again, we split the proof into two steps. In the first step, we show that this surface is closed, convex, and sufficiently smooth. In the second step, we compute the umbilic points and their indices.

Step 1. We first show the surface is regular. This surface is expressible as $f^{-1}(0)$, where

$$f(x, y, z) = \frac{x^2}{4} + \epsilon x^4 + \frac{y^2}{4} + \epsilon y^4 + \frac{z^2}{9} - 1.$$

f is infinitely differentiable. Its partial derivatives $f_x = \frac{x}{2} + 4\epsilon x^3$, $f_y = \frac{y}{2} + 4\epsilon y^3$, $f_z = \frac{2z}{9}$ vanish simultaneously only at $(x, y, z) = (0, 0, 0)$, which is not contained in $f^{-1}(0)$. Thus, 0 is a regular value of f, and this surface is regular. To prove this surface is convex, we compute its Gaussian curvature K.

$$K = \frac{eg - f^2}{EG - F^2}$$

Since $-4\epsilon u^4 - 4\epsilon v^4 - u^2 - v^2 + 4 > 0$, we know that $\mathrm{sgn}(eg - f^2) = \mathrm{sgn}(u^2(8\epsilon u^2+1)^2(24\epsilon v^2 + 1) + v^2(8\epsilon v^2+1)^2(24\epsilon u^2+1) + (24\epsilon u^2+1)(24\epsilon v^2+1)(-4\epsilon u^4 - 4\epsilon v^4 - u^2 - v^2 + 4)) = 1$. Also, $\mathrm{sgn}(EG - F^2) = \mathrm{sgn}(4(-4\epsilon u^4 - 4\epsilon v^4 - u^2 - v^2 + 4) + 9v^2(8\epsilon v^2+1)^2 + 9u^2(8\epsilon u^2+1)^2) = 1$. So, $\mathrm{sgn}(K) = 1$ and this surface is convex.

Step 2. We use the same method as in the first surface. Refer to appendix [2] for detailed calculations of the fundamental form coefficients. $Gf = Fg$ becomes

$$(16Q + 9v^2(8\epsilon v^2+1)^2)(-\frac{3}{16}uv(8\epsilon u^2+1)(8\epsilon v^2+1)Q^{-\frac{3}{2}}) =$$
$$(9uv(8\epsilon u^2+1)(8\epsilon v^2+1))(-\frac{3}{4}\left(\frac{1}{4}v^2(8\epsilon v^2+1)^2Q^{-\frac{3}{2}} + (24\epsilon v^2+1)Q^{-\frac{1}{2}}\right)) \Rightarrow$$
$$-\frac{3}{16}uv(8\epsilon u^2+1)(8\epsilon v^2+1)Q^{-\frac{3}{2}}(16Q + 9v^2(8\epsilon v^2+1)^2 - 9\cdot(v^2(8\epsilon v^2+1)^2 + (24\epsilon v^2+1)4Q)) = 0 \Rightarrow$$
$$\frac{3}{4}uv(8\epsilon u^2+1)(8\epsilon v^2+1)Q^{-\frac{1}{2}}(5 + 216\epsilon v^2) = 0$$

Similarly, $Ef = eF$ becomes

$$\frac{3}{4}uv(8\epsilon u^2+1)(8\epsilon v^2+1)Q^{-\frac{1}{2}}(5 + 216\epsilon u^2) = 0$$

$Eg = Ge$ becomes

$$(16Q + 9u^2(8\epsilon u^2+1)^2)(-\frac{3}{4}\left(\frac{1}{4}v^2(8\epsilon v^2+1)^2Q^{-\frac{3}{2}} + (24\epsilon v^2+1)Q^{-\frac{1}{2}}\right)) =$$
$$(16Q + 9v^2(8\epsilon v^2+1)^2)((-\frac{3}{4}\left(\frac{1}{4}u^2(8\epsilon u^2+1)^2Q^{-\frac{3}{2}} + (24\epsilon u^2+1)Q^{-\frac{1}{2}}\right)) \Rightarrow$$
$$(16Q + 9u^2(8\epsilon u^2+1)^2)\left(v^2(8\epsilon v^2+1)^2 + (24\epsilon v^2+1)4Q\right) =$$
$$(16Q + 9v^2(8\epsilon v^2+1)^2)\left(u^2(8\epsilon u^2+1)^2 + (24\epsilon u^2+1)4Q\right).$$

Solving these three equations simultaneously gives the solutions

$$(u, v) = (0, 0), (0, \pm\sqrt{\tfrac{\sqrt{64\epsilon+1}-1}{8\epsilon}}), (\pm\sqrt{\tfrac{\sqrt{64\epsilon+1}-1}{8\epsilon}}, 0)$$

However, the latter two pairs of solutions are on the boundary of the parameterization $\frac{x^2}{4} + \epsilon x^4 + \frac{y^2}{4} + \epsilon y^4 = 1$ on which the fundamental form coefficients are not defined. Thus, to check this boundary for umbilic points, we consider the new rotated surface $\frac{x^2}{4} + \epsilon x^4 + \frac{y}{9} + \frac{z^2}{4} + \epsilon z^4 = 1$ parameterized by $\mathcal{S} = \left(u, v, \sqrt{\frac{-1+\sqrt{1+64\epsilon(1-\frac{u^2}{4}-\epsilon u^4-\frac{v^2}{9})}}{8\epsilon}}\right)$. Since we only need to check the boundary of our previous parameterization for umbilic points, we can substitute $y = 0$ in all computations. Refer to appendix

[3] for detailed computations of the fundamental form coefficients. We check that these coefficients are defined as long as $R-1>0 \Rightarrow 1-\frac{x^2}{4}-\epsilon x^4-\frac{y^2}{9}>0$, which is always true. Note that after substitution, $F=f=0$, so that $Gf=Fg$ and $Ef=Fe$ are always satisfied. $Eg=Ge$ then becomes

$$x^2\left(2\epsilon+\frac{1}{2}\right)^2(432R-352)+R^2\left(\frac{R-1}{\epsilon}\right)\left(18\epsilon+\frac{5}{2}\right)=0.$$

Note that for $R>1$, the quantity on the left must be nonnegative and the quantity on the right must be positive, so this solution has no equations. Thus, no umbilic points exist on the boundary of the first parameterization, and we then conclude that this surface has only 2 umbilic points. By the Poincaré-Hopf index theorem, each of these 2 umbilic points has index 1. □

Remark 3.1. It is interesting here how neither the umbilic points nor their indices depend on ϵ. We can get arbitrarily close to the known example of the rotationally symmetric ellipsoid and the locations and indices of the umbilic points will remain constant.

Future Work

We would like to prove or disprove Proposition 1.1:

1. For surfaces of the form $ax^{2k}+by^{2k}+cz^{2k}=1$ with $a,b,c>0, k\in\mathbb{Z}_{>1}$, are the indices of the umbilic points independent of a,b,c,k?

Here, we reach technical limitations when plugging the DE in Lemma 1.3 into Mathematica. Its approximation method NDSolve does not accept symbolic variables such as k, and its exact method DSolve requires an infeasibly long running time.

In this paper, we have explored the umbilic points and their indices of the surface $\frac{x^2}{4}+\epsilon x^4+\frac{y^2}{4}+\epsilon y^4+\frac{z^2}{9}=1$. In the future, we would like to generalize this surface to the example $ax^2+\epsilon x^4+ay^2+\epsilon y^4+cz^2=1$. To that end, we pose the following question:

2. What are the locations and indices of the umbilic points of the surface $ax^2+\epsilon x^4+ay^2+\epsilon y^4+cz^2=1$ with $a,\epsilon,c>0$?

We suspect that, like the example we have found, this surface will have two umbilic points at $\left(0,0,\pm\sqrt{\frac{1}{c}}\right)$ each with index 1.

Appendix

[1]

We include the computation of the fundamental form coefficients below. Let $\mathcal{S}=(u,v,(1-au^{2k}-bv^{2k})^{\frac{1}{2k}}$. Then,

$$\vec{\mathcal{S}}_u=(1,0,-ac^{-\frac{1}{2k}}u^{2k-1}(-au^{2k}-bv^{2k}+1)^{\frac{1}{2k}-1}),$$
$$\vec{\mathcal{S}}_v=(0,1,-bc^{-\frac{1}{2k}}u^{2k-1}(-au^{2k}-bv^{2k}+1)^{\frac{1}{2k}-1}),$$
$$\vec{\mathcal{S}}_{uu}=\left(0,0,a(2k-1)c^{-\frac{1}{2k}}u^{2k-2}(bv^{2k}-1)(-au^{2k}-bv^{2k}+1)^{-\frac{1}{2k}-2}\right),$$

$$\begin{aligned}
\vec{\mathcal{S}}_{uv} &= \left(0, 0, 2abk(\frac{1}{2k}-1)c^{-\frac{1}{2k}}x^{2k-1}y^{2k-1}(-ax^{2k}-by^{2k}+1)^{\frac{1}{2k}-2}\right),\\
\vec{\mathcal{S}}_{vv} &= \left(0, 0, b(2k-1)c^{-\frac{1}{2k}}x^{2k-2}(ax^{2k}-1)(-ax^{2k}-by^{2k}+1)^{\frac{1}{2k}-2}\right),\\
E = \vec{\mathcal{S}}_u \cdot \vec{\mathcal{S}}_u &= 1 + a^2c^{-\frac{1}{k}}x^{4k-2}(-au^{2k}-bv^{2k}+1)^{\frac{1}{k}-2},\\
F = \vec{\mathcal{S}}_u \cdot \vec{\mathcal{S}}_v &= abc^{-\frac{1}{k}}x^{2k-1}y^{2k-1}(-ax^{2k}-by^{2k}+1)^{\frac{1}{k}-2},\\
G = \vec{\mathcal{S}}_v \cdot \vec{\mathcal{S}}_v &= 1 + b^2c^{-\frac{1}{k}}v^{4k-2}(-au^{2k}-bv^{2k}+1)^{\frac{1}{k}-2}.
\end{aligned}$$

Also,

$$\vec{N} = \frac{\vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v}{\mid \vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v \mid} = \frac{(ac^{-\frac{1}{2k}}x^{2k-1}(-ax^{2k}-by^{2k}+1)^{\frac{1}{2k}-1}, bc^{-\frac{1}{2k}}y^{2k-1}(-ax^{2k}-by^{2k}+1)^{\frac{1}{2k}-1}, 1)}{\sqrt{(-ax^{2k}-by^{2k}+1)^{\frac{1}{k}-2}(a^2c^{-\frac{1}{k}}x^{4k-2}+b^2c^{-\frac{1}{k}}y^{4k-2})+1}},$$

$$e = -\vec{N}\cdot\vec{\mathcal{S}}_{uu} = -\frac{\mid \vec{\mathcal{S}}_{uu} \mid}{\mid \vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v \mid}, \quad f = -\vec{N}\cdot\vec{\mathcal{S}}_{uv} = -\frac{\mid \vec{\mathcal{S}}_{uv} \mid}{\mid \vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v \mid}, \quad g = -\vec{N}\cdot\vec{\mathcal{S}}_{vv} = -\frac{\mid \vec{\mathcal{S}}_{vv} \mid}{\mid \vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v \mid}.$$

[**2**]

We include the computation of the fundamental form coefficients below. Let $\mathcal{S} = (u, v, \sqrt{9(1-\epsilon u^4 - \frac{u^2}{4} - \epsilon v^4 - \frac{v^2}{4}})$. Then,

$$\begin{aligned}
\vec{\mathcal{S}}_u &= (1, 0, -\frac{3}{2}u(8\epsilon u^2+1)(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)^{-\frac{1}{2}})\\
\vec{\mathcal{S}}_v &= (0, 1, -\frac{3}{2}v(8\epsilon v^2+1)(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)^{-\frac{1}{2}})\\
\vec{\mathcal{S}}_{uu} &= \left(0, 0, -\frac{3}{2}\left(\frac{u^2(8\epsilon u^2+1)^2+(24\epsilon u^2+1)(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)}{(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)^{\frac{3}{2}}}\right)\right)\\
\vec{\mathcal{S}}_{uv} &= \left(0, 0, -\frac{3}{2}\left(\frac{uv(8\epsilon u^2+1)(8\epsilon v^2+1)}{(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)^{\frac{3}{2}}}\right)\right)\\
\vec{\mathcal{S}}_{vv} &= \left(0, 0, -\frac{3}{2}\left(\frac{v^2(8\epsilon v^2+1)^2+(24\epsilon v^2+1)(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)}{(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)^{\frac{3}{2}}}\right)\right)\\
E = \vec{\mathcal{S}}_u \cdot \vec{\mathcal{S}}_u &= 1 + \frac{9u^2(8\epsilon u^2+1)^2}{4(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)}\\
F = \vec{\mathcal{S}}_u \cdot \vec{\mathcal{S}}_v &= \frac{9uv(8\epsilon u^2+1)(8\epsilon v^2+1)}{4(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)}\\
G = \vec{\mathcal{S}}_u \cdot \vec{\mathcal{S}}_u &= 1 + \frac{9v^2(8\epsilon v^2+1)^2}{4(-4\epsilon u^4-4\epsilon v^4-u^2-v^2+4)}
\end{aligned}$$

Also,

$$\vec{N} = \frac{\vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v}{\mid \vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v \mid} = \frac{(\frac{3}{4}u(8\epsilon u^2+1)Q^{-\frac{1}{2}}, \frac{3}{4}v(8\epsilon v^2+1)Q^{-\frac{1}{2}}, 1)}{\sqrt{\frac{9}{16}u^2(8\epsilon u^2+1)^2Q^{-1}+\frac{9}{16}v^2(8\epsilon v^2+1)^2Q^{-1}+1}}$$

$$e = -\vec{N}\cdot\vec{\mathcal{S}}_{uu} = -\frac{\mid \vec{\mathcal{S}}_{uu} \mid}{\mid \vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v \mid}, \quad f = -\vec{N}\cdot\vec{\mathcal{S}}_{uv} = -\frac{\mid \vec{\mathcal{S}}_{uv} \mid}{\mid \vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v \mid}, \quad g = -\vec{N}\cdot\vec{\mathcal{S}}_{vv} = -\frac{\mid \vec{\mathcal{S}}_{vv} \mid}{\mid \vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v \mid}.$$

[**3**]

We include the computation of the fundamental form coefficients below. For the sake of simplicity, we replace $\sqrt{1+64\epsilon(1-\frac{u^2}{4}-\epsilon u^4-\frac{v^2}{9})}$ by R.

$$\vec{\mathcal{S}}_u = \left(1, 0, \frac{4\sqrt{2}(-4\epsilon u^3 - \frac{u}{2})}{R\sqrt{\frac{R-1}{\epsilon}}}\right)$$

$$\vec{\mathcal{S}}_v = \left(0, 1, \frac{-8\sqrt{2}y}{9R\sqrt{\frac{R-1}{\epsilon}}}\right) = (0,1,0)$$

$$\vec{\mathcal{S}}_{uu} = \left(0, 0, \frac{1}{2\sqrt{2}}\left(-\frac{512\epsilon(-4\epsilon x^3 - \frac{x}{2})^2}{R^3\sqrt{\frac{R-1}{\epsilon}}} - \frac{256(-4\epsilon x^3 - \frac{x}{2})^2}{R^2(\frac{R-1}{\epsilon})^{\frac{3}{2}}} + \frac{16(-12\epsilon u^2 - \frac{1}{2})}{R\sqrt{\frac{R-1}{\epsilon}}}\right)\right)$$

$$\vec{\mathcal{S}}_{uv} = \left(0, 0, \frac{256\sqrt{2}\epsilon y(-4\epsilon x^3 - \frac{x}{2})}{9R^3\sqrt{\frac{R-1}{\epsilon}}} + \frac{128\sqrt{2}y(-4\epsilon x^3 - \frac{x}{2})}{9R^2(\frac{R-1}{\epsilon})^{\frac{3}{2}}}\right) = (0,0,0)$$

$$\vec{\mathcal{S}}_{vv} = \left(0, 0, \frac{1}{2\sqrt{2}}\left(-\frac{2048\epsilon y^2}{81R^3\sqrt{\frac{R-1}{\epsilon}}} - \frac{1024y^2}{81R^2(\frac{R-1}{\epsilon})^{\frac{3}{2}}} - \frac{32}{9R\sqrt{\frac{R-1}{\epsilon}}}\right)\right) = \left(0, 0, -\frac{16}{9\sqrt{2}R\sqrt{\frac{R-1}{\epsilon}}}\right)$$

$$E = \vec{\mathcal{S}}_u \cdot \vec{\mathcal{S}}_u = 1 + \frac{32(-4\epsilon u^3 - \frac{u}{2})^2}{R^2(\frac{R-1}{\epsilon})}$$

$$F = \vec{\mathcal{S}}_u \cdot \vec{\mathcal{S}}_v = \frac{-64y(-4\epsilon u^3 - \frac{u}{2})}{9R^2(\frac{R-1}{\epsilon})} = 0$$

$$G = \vec{\mathcal{S}}_v \cdot \vec{\mathcal{S}}_v = 1 + \frac{128y^2}{81R^2(\frac{R-1}{\epsilon})} = 1$$

Also,

$$\vec{N} = \frac{\vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v}{|\ \vec{\mathcal{S}}_u \times \vec{\mathcal{S}}_v\ |} = \frac{(-\frac{4\sqrt{2}(-4\epsilon u^3 - \frac{u}{2})}{R\sqrt{\frac{R-1}{\epsilon}}}, \frac{8\sqrt{2}y}{9R\sqrt{\frac{R-1}{\epsilon}}}, 1)}{\sqrt{\frac{32(-2\epsilon u - \frac{u}{2})^2 \cdot 81 + 128y^2}{81R^2(\frac{R-1}{\epsilon})} + 1}} = \frac{(-\frac{4\sqrt{2}(-4\epsilon u^3 - \frac{u}{2})}{R\sqrt{\frac{R-1}{\epsilon}}}, 0, 1)}{\sqrt{\frac{32(-2\epsilon u - \frac{u}{2})^2}{R^2(\frac{R-1}{\epsilon})} + 1}}$$

$$e = -\vec{N} \cdot \vec{\mathcal{S}}_{uu}, \qquad f = -\vec{N} \cdot \vec{\mathcal{S}}_{uv} = 0, \qquad g = -\vec{N} \cdot \vec{\mathcal{S}}_{vv}$$

[**4**]

We overlay several solution curves for the following tested values of a, b, c, k. Note that they all have the same general shape, even when a, b, c, or k is relatively large, suggesting a fixed index. Taking advantage of symmetry gives Figure 6.

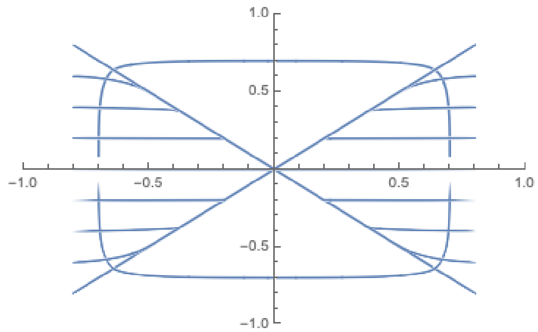


Figure 10: $a = 1, b = 1, c = 1, k = 2$

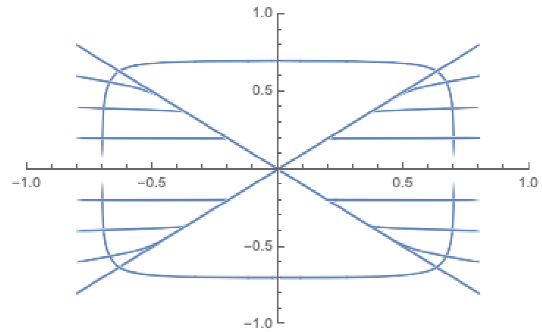


Figure 11: $a = 1, b = 1, c = 100, k = 2$

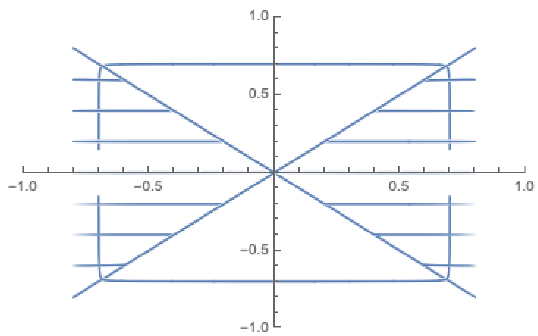


Figure 12: $a = 1, b = 1, c = 1, k = 4$

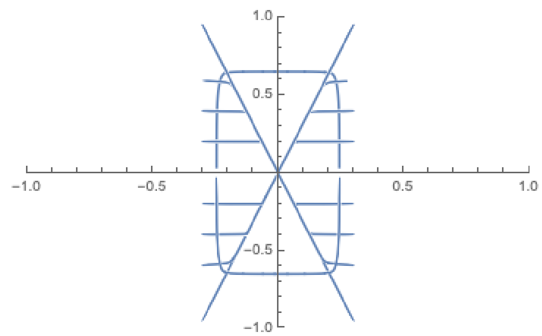


Figure 13: $a = 1, b = 10, c = 10, k = 2$

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Credits

Professor Wu has proposed the research topic, suggested studying the second surface, and given reading references. The author has done background reading, found the first surface, and created and performed all computation methods (except the method for calculating fundamental form coefficients, which is well known).

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