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参赛队员姓名: Team Member Name:	**Espen Slettnes**
中学: High School:	**Stanford Online High School**
省份: Province:	**California**
国家/地区: Country/Region:	**USA/North America**
指导教师姓名: Mentor Name:	**Dr. Jesse Geneson** **Iowa State University** **(MIT PRIMES-USA research mentor)**
论文题目: Essay Topic:	**Minimal Embedding Dimensions of Rectangle *k*-Visibility Graphs**

Minimal Embedding Dimensions of Rectangle k-Visiblity Graphs

Espen Slettnes*

MIT PRIMES-USA, United States

September 1, 2019

Abstract

Bar visibility graphs were adopted in the 1980s as a model to represent traces, e.g., on circuit boards and in VLSI chip designs. Two generalizations of bar visibility graphs, rectangle visibility graphs and bar k-visibility graphs, were subsequently introduced.

Here, we combine bar k- and rectangle visibility graphs to form rectangle k-visibility graphs (RkVGs), and further generalize these to higher dimensions. A graph is a d-dimensional RkVG if and only if it can be represented with vertices as disjoint axis-aligned hyperrectangles in d-space, such that there is an axis-parallel line of sight between two hyperrectangles that intersects at most k other hyperrectangles if and only if there is an edge between the two corresponding vertices.

For any graph G and a fixed k, we prove that given enough spacial dimensions, G has a rectangle k-visibility representation, and thus we define the minimal embedding dimension (MED) with k-visibility of G to be the smallest d such that G is a d-dimensional RkVG. We study the properties of MEDs and find upper bounds on the MEDs of various types of graphs. In particular, we find that the k-visibility MED of the complete graph on m vertices K_m is at most $m/(2(k+1))$, of complete r-partite graphs is at most $r+1$, and of the m^{th} hypercube graph Q_m is at most $\lceil 2m/3 \rceil$ in general, and at most $\lfloor \sqrt{m} \rceil$ for $k=0,\ m \neq 2$.

Keywords: combinatorics, graphs, graph theory, graph applications, extremal combinatorics, visibility, visibility graph, bar visibility, rectangle visiblity, unit bar visibility, unit rectangle visibility, bar k-visibility, interval graph, Gray code, Cartesian product of graphs, rooted product of graphs, corona product of graphs

*Email address: espen@slett.net

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1 Introduction

Bar visibility graphs were introduced in the 1980s as a way to model circuit traces in VLSI chip designs (Lodi and Pagli, [12]). A graph G is a bar visibility graph if there is a one-to-one correspondence between each of its vertices and horizontal bars, such that there is an unobstructed vertical line of sight between two bars (i.e., a vertical line segment between the two bars not intersecting other bars) if and only if there is an edge between the corresponding vertices in G. Note that the bars and visibility lines form a planar graph drawing of G.

In their 1997 paper *On Rectangle Visibility Graphs* [2], Bose et al. introduced rectangle visibility graphs as "a graph in the plane so that the vertices of the graph are rectangles that are aligned with the axes, and the edges of the graph are horizontal or vertical lines-of-sight". Previously, though using different terminology, Stephen Wismath established in his 1989 thesis [15] that all planar graphs are rectangle visibility graphs (i.e., have rectangle visibility representations).

Dean et al. introduced bar k-visibility graphs in 2007 [6] as a generalization in which the visibility line between the bars are relaxed from being unobstructed to being obstructed by at most k other bars. Hartke et al. published *Further Results on Bar k-Visibility Graphs* [11], and in combination these two papers established that the maximum number of edges in a bar k-visibility graph on n vertices is $(k+1)(3n-4k-6)$. Dean et al. further proved that the thickness of every bar 1-visibility graph is at most 4, and Chang et al. [3] proved that the thickness of a bar k-visibility graph is at most $3k+3$.

Others, such as Babbit et al. [1], have studied k-visibility on other types of visibility representations. Here we define a *rectangle k-visibility graph* to be a graph that can be represented with vertices as disjoint axis-parallel rectangles, where there is an edge between two vertices if and only if there is an axis-parallel lines of sight between the corresponding rectangles which is obstructed by at most k other rectangles. By the above, as edges corresponding to horizontal as well as vertical visibility lines form bar k-visibility graphs, the number of edges and the thickness in such a graph are at most $2(k+1)(3n-4k-6)$ and $6k+6$, respectively. In particular, the respective thickness of rectangle 0- and 1-visibility graphs are at most 2 and 8.

Prior research has further generalized rectangle visibility graphs into 3 dimensions, where they are referred to as *box* visibility graphs [8]. Here we consider a generalization of rectangle k-visibility graphs into higher dimensions, and in particular, study the minimum dimension needed to represent various graphs with k-visibility for a fixed k. For example, as discussed above, the minimal embedding dimension (MED) of a planar graph given $k=0$ is at most 2.

We study such MEDs on general graphs in Section 3. Among other things, we show that the MED of a connected graph G on n vertices is at most $\lceil \frac{n}{2} \rceil$, that the MED of a disconnected graph G is the maximum of 2 and the MEDs of its connected components, and that MEDs are subadditive under the Cartesian product.

We then move on to specific graphs. We cover complete graphs in Section 4, where we establish that MEDs can be arbitrarily large and that the MED of the complete graph on m vertices K_m is at most $\max\left(\left\lceil \frac{m-22(\lfloor k/2 \rfloor+1)}{2(k+1)} \right\rceil + 1, 3\right)$. In Section 5 on multipartite graphs we find that the MED of a complete r-partite graph is at most $r+1$. Finally, Section 6 is devoted to hypercubes; we show that the MED of Q_m is at most $\lceil \frac{2}{3}m \rceil$, and at most $\lfloor \sqrt{m} \rceil$ for $k=0, m \neq 2$.

2 Terminology

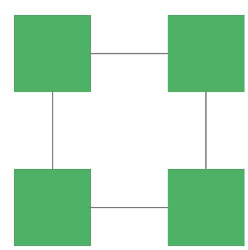
We define a d-dimensional *rectangle visibility graph* (RVG^s) to be a graph where vertices can be represented as (closed hyper-)rectangles in d dimensions, and edges as all axis-parallel lines of sight between (i.e., unobstructed line segments connecting) these (hyper-)rectangles. We also define the following variants:

- An $\epsilon-$*visibility graph* (RVG$^\epsilon$) imposes a positive thickness to the line of sight between rectangles, such that the rectangles must overlap by a positive amount in all $(d-1)$ orthogonal dimensions of the line of sight. In contrast, a *strong visibility graph* (RVG^s) allows visibility lines with zero thickness, thus zero overlap in orthogonal dimensions.
- A *rectangle k-visibility graph* (RkVG^s, RkVG$^\epsilon$) allows the line of sight to be obstructed by up to k other rectangles.
- A *unit rectangle visibility graph* (URVG^s, URVG$^\epsilon$, URkVG^s, URkVG$^\epsilon$) imposes the restriction that all (hyper-)rectangles have the same dimensions (typically unit hypercubes).

Unless explicitly stated, we use the term *rectangle* to mean *d-dimensional hyper-rectangle*. As a special case, a *box* is a 3-dimensional rectangle.

The *minimal embedding dimension* (MED) of a graph G is the smallest number of spacial dimensions d for the graph to be a specific one of the above. We denote by $M^s(G), \mu^s(G), M^s_k(G), \mu^s_k(G), M^\epsilon(G), \mu^\epsilon(G), M^\epsilon_k(G)$ and $\mu^\epsilon_k(G)$ the MEDs of G as a RVG^s, URVG^s, RkVG^s, URkVG^s, RVG$^\epsilon$, URVG$^\epsilon$, RkVG$^\epsilon$, and URkVG$^\epsilon$, respectively.

Example 1. $\mu^s_1(C_4) = 2$ is the smallest number of dimensions in which we can represent C_4 as a unit rectangle 1-visibility graph with strong visibility.



Additionally, we use the following conventions:

- G will be a graph. (We do not consider the null graph on zero vertices.)
- $n := |V(G)| \geq 1$ is the number of vertices (i.e., size) of G.
- The $^\epsilon$ or ^{s} suffix may be omitted, in which case the statement applies to both strong or $\epsilon-$visibility.

Example 2. "G is an $M(G)$-dimensional RVG" means that "G is an $M^s(G)$-dimensional RVG^s and G is an $M^\epsilon(G)$-dimensional RVG$^\epsilon$".

- All occurrences of $\left[\begin{smallmatrix}\mu \\ M\end{smallmatrix}\right]$ can be consistently replaced by either μ or M.

  Example 3. "G is an $\left[\begin{smallmatrix}\mu \\ M\end{smallmatrix}\right](G)$-dimensional (U)RVG" means that "G is an $\mu(G)$-dimensional URVG and G is an $M(G)$-dimensional RVG".

3 General Graphs

3.1 Existence of the minimal embedding dimensions

Here we will prove that the minimal embedding dimension is well-defined, i.e. that every graph has a minimal embedding dimension. To that end, we first show how to think of a representation of a d-dimensional (U)RkVG in terms of its projections to the axes.

Definition 4. A graph G is an *interval graph* if there is a one-to-one correspondence between its vertices and a set of (closed) intervals, such that two intervals overlap if and only if there is an edge between the corresponding vertices in G.

A *unit interval graph*, more commonly known as an *indifference graph*, is an interval graph that can be represented with unit intervals.

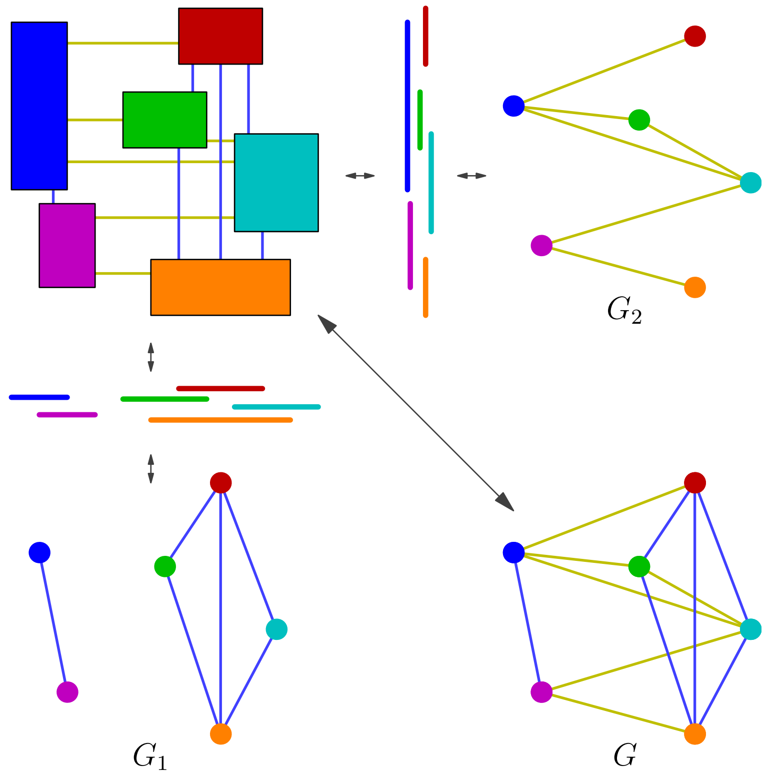


Figure 1: A graph G represented as a 2-dimensional RkVG, with projected intervals in each dimension corresponding to vertices in interval graphs G_1 and G_2

Lemma 5. *A graph G with n vertices is a d-dimensional (U)RkVG^s, where $k \geq n-2$, if and only if there exist d (unit) interval graphs $G_1, \ldots, G_d$ of G on the same vertex set as G such that no edge is contained in all of $G_1, \ldots, G_d$ and two vertices $u, v \in G$ are adjacent if and only if they are adjacent in all but exactly one of $G_1, \ldots, G_d$.*

Proof. We note that $k \geq n-2$ is the same as "infinite" visibility, as at most $n-2$ rectangles can obstruct a visibility line between any two rectangles.

First we go from a d-dimensional (U)RkVG^s G to corresponding (unit) interval graphs $G_1, \ldots, G_d$.

Consider the projections of all rectangles onto each of the axes of $\mathbb{R}^d$. Let G_i be the (unit) interval graph formed by the projection onto the i^{th} axis. Two rectangles cannot overlap in all of these projections, lest they would themselves overlap. In other words, no edge can be in all of $G_1, \ldots, G_d$.

Iff two rectangles can see each other via a visibility line in the direction of the i^{th} axis $(1 \leq i \leq d)$, their respective projections do not overlap on the i^{th} axis, but overlap on all other axes $j \mid (1 \leq j \leq d)$. In other words, two vertices G are adjacent if and only if they are adjacent in all but exactly one of $G_1, \ldots, G_d$.

Then to reconstruct the original rectangles if we have a set of (unit) interval graphs $G_1, \ldots, G_d$, we can simply take the arrangement of rectangles for which the (unit) rectangle projections onto the axes correspond to the (unit) interval representations of G_1 through G_d. $\square$

With this in mind, we now construct a representation of any graph G as a (U)RkVG by specifying its projections.

Theorem 6. *Every graph has a minimal embedding dimension as a (U)RkVG. Specifically, for a graph G on n vertices, $\left[\begin{smallmatrix}\mu\\ M\end{smallmatrix}\right]_k(G) \leq n$.*

Proof. Let G's vertex set be $[n] = \{1, 2, \ldots, n\})$, and let

$$S_i(u) = \begin{cases} [0,1] & \text{if } u = i \\ \left[\frac{2}{3}, \frac{5}{3}\right] & \text{if } (i \not\sim_G u) \vee (u < i) \\ \left[\frac{4}{3}, \frac{7}{3}\right] & \text{if } (i \sim_G u) \wedge (u > i) \end{cases}$$

for $i, u \in [n]$. ("$\sim_G$" denotes the adjacency relation in G.)

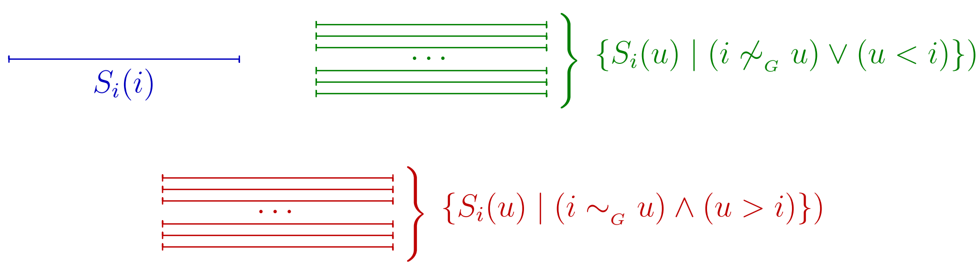


Figure 2: The n unit intervals $\{S_i(v) \mid v \in [n]\})$ (with artificial elevations added for illustration)

Let G_i be the (unit) interval graph formed by S_i. Note that

(*a*) In S_i's range, there is no interval strictly between two other intervals,
(*b*) any two intervals $S_i(u), S_i(v) \mid u, v \neq i$ overlap,
(*c*) if $u \not\sim v$, intervals $S_i(u)$ and $S_i(v)$ do not overlap for $i \in \{u, v\})$,
(*d*) if $u \sim v \wedge u < v$, intervals $S_i(u)$ and $S_i(v)$ overlap for $i = u$ but not for $i = v, d$ and
(*e*) all overlaps are positive.

By (*a*), no rectangle can block a visibility line between two others, and by (*e*), strong vs. ϵ-visibility doesn't matter, so we can use Lemma 5.

By (*c*) and (*d*), no edge is in all of the G_i's. By (*b*) and (*d*), if $u \sim v$, they are adjacent in all but one G_i representation. Finally, by (*c*), if $u \not\sim v$, they are not adjacent in two G_i's.

Thus, by Lemma 5, G is a d-dimensional (U)RkVG. $\square$

3.2 Basic Properties

We now make the following observations about minimal embedding dimensions:

Lemma 7. *Given a graph G on n vertices, $M_k(G) \leq \mu_k(G)$.*

Proof. Any representation of G as a URkVG in $\mu_k(G)$ dimensions is also a valid representation of G as RkVG, thus $M_k(G) \leq \mu_k(G)$. □

Lemma 8. *Given a graph G on n vertices, $\left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k^\epsilon(G) \leq \left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k^s(G)$.*

Proof. Given a representation of G as a (U)RkVG, let δ_i be the smallest nonzero difference between coordinates along the i^{th} of sides of any two of its hyperrectangles. Expand the rectangles by a margin of $\frac{\delta_i}{3}$ in each dimension i.

No new visibility lines have been created or destroyed, as the overlaps between the rectangles have not changed in any dimension. Moreover, any two rectangles that previously had any overlap now have a positive overlap. Thus, we now have a representation of G as a (U)RkVG$^\epsilon$, and $\left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k^\epsilon(G) \leq \left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k^s(G)$ as desired. □

Lemma 9. *A graph G is a d-dimensional (U)RkVG if and only if $d \geq \left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k(G)$.*

Proof. The former implies the latter by definition.

The latter implies the former because we can take a representation of G in $M_k(G)$ dimensions, place it in d-dimensional space, and thicken it by 1 unit in the remaining $(d - \left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k(G))$ dimensions. □

3.3 Minimal Embedding Dimensions of General Graphs

Theorem 10. *Let G be a connected graph on n vertices. Then, $M_k(G) \leq \left\lceil \frac{n}{2} \right\rceil$.*

Proof. Let $S = \{v_1, \ldots, v_n\})$ be the vertices of G, where, WLOG, v_n shares an edge with v_{n-1} if $n \geq 2$. We divide S into subsets of at most 4 vertices, such that $S_m = \left\{v_{4m-3}, \ldots, v_{\min(n,4m)}\right\})$ for $m \in \left[1, \left\lceil \frac{n}{4} \right\rceil\right]$. Let G_m be the induced subgraph formed by vertices in S_m. Note that if $\left|S_{\lceil \frac{n}{4} \rceil}\right| \geq 2$, $G_{\lceil \frac{n}{4} \rceil}$ has at least one edge.

Let $\mathcal{S}_1, \ldots, \mathcal{S}_{\lceil \frac{n-2}{4} \rceil}$ be orthogonal 2-dimensional spaces, and if $n \equiv 1 \pmod 4 \vee n \equiv 2 \pmod 4$, let $\mathcal{S}_{\lceil \frac{n}{4} \rceil}$ be an additional orthogonal 1-dimensional space. We will construct a rectangle visibility representation of G by constructing its projections onto these spaces.

The projection of S_m onto $\mathcal{S}_m$ will be one of the arrangements in Figure 3, such that the visibility graph formed between the green rectangles is G_m (all possible values of G_m are covered in Figure 3).

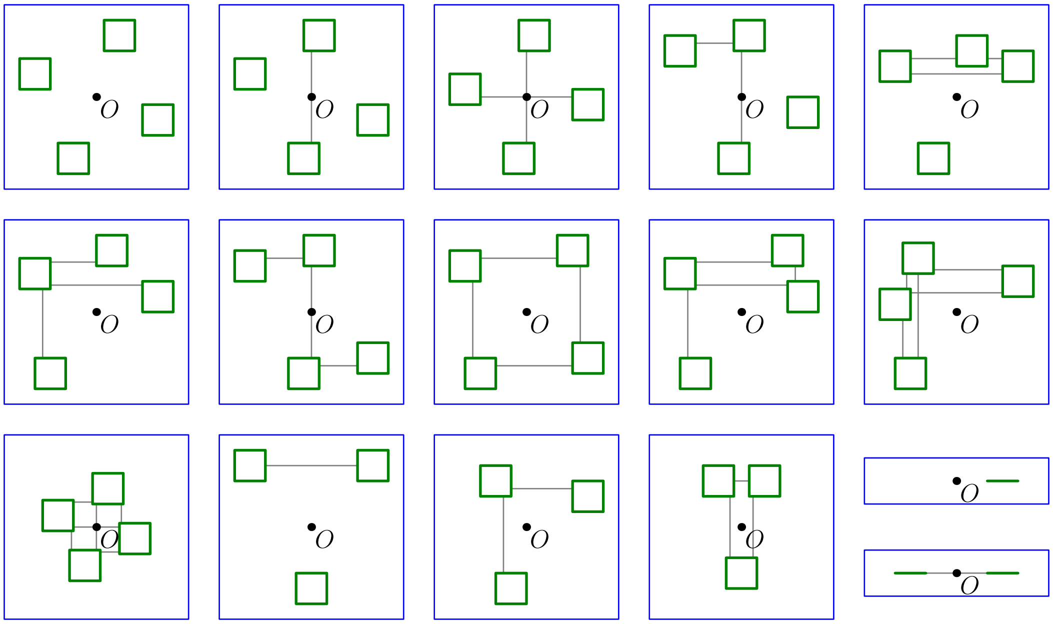


Figure 3: All possible projections of vertices $v_{4m-3}, \ldots, v_{\min(4m,n)}$ into $\mathcal{S}_m$

Let T_m be the complimentary set $S \setminus S_m$. We project every other vertex $v_i \in T_m$ onto the same 2-dimensional subspace $\mathcal{S}_m$ in such a way that each projection covers the central point O, and either overlaps or is adjacent to each vertex in $v_j \in S_m$. Some possible projections are illustrated in Figure 4.

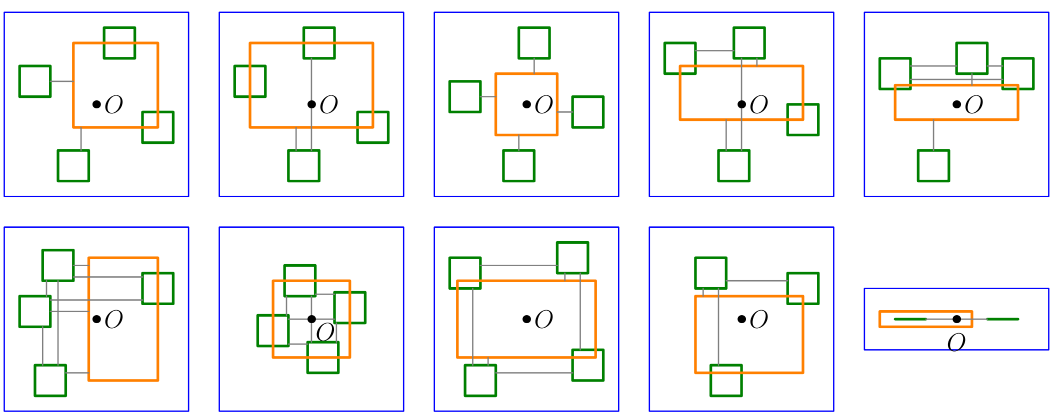


Figure 4: Sample projections of an additional vertex v_i, overlapping a central point O and either overlapping or adjacent to each of $v_{4m-3}, \ldots, v_{\min(4m,n)}$

We use the following rules:

- If $i < j$, the projections of v_i and v_j will *not* overlap; this counts as being disjoint in one dimension.
- If $i > j$, the projections of v_i and v_j will overlap if and only if $v_i \sim v_j$. If not, this counts as

being disjoint in a second dimension, thus precluding any axis-parallel visibility line between the corresponding rectangles.

We note that every vertex $v_i \in T_m$ overlaps with point O in $\mathcal{S}_m$, thus there are no more disjoint projections than those described here.

By construction, we now have a representation of G where all pairs of vertices (v_i, v_j) are disjoint in one dimension if they are adjacent, and in two dimensions if they are non-adjacent. Moreover, there does not exist any third vertex v_k that blocks visibility between v_i and v_j (in particular in $\mathcal{S}_{\lceil \frac{k}{4} \rceil}$ at O for $k \notin S_i, S_j$), so no rectangle can block a visibility line. By Lemma 5, this means that we have a valid representation of G in $\lceil \frac{n}{2} \rceil$ dimensions. □

3.4 Graph Composition

We now look at relationships between the MEDs and various graph products.

3.4.1 Disjoint Union

We find the minimal embedding dimensions of the disjoint union of two graphs:

Lemma 11. *Let* G_1, G_2 *be graphs with disjoint vertex sets, and* $D = \max\left(\left[{\mu \atop M}\right]_k(G_1), \left[{\mu \atop M}\right]_k(G_2)\right)$.

If $D \geq 2$*, the minimal embedding dimension of their disjoint union is* $\left[{\mu \atop M}\right]_k(G_1 \sqcup G_2) = D$.

Proof. We will separately prove that $\left[{\mu \atop M}\right]_k(G_1 \sqcup G_2) \leq D$ and that $\left[{\mu \atop M}\right]_k(G_1 \sqcup G_2) \geq D$.

$\underline{\left[{\mu \atop M}\right]_k(G_1 \sqcup G_2) \leq D}$

By Lemma 9, representations exist for each of G_1 and G_2 in D dimensions. By placing both of these representations in the same D-space in such a way that that they are non-overlapping in at least 2 dimensions, i.e., "diagonally", we ensure that there exists no visibility lines between any vertex in G_1 and any vertex in G_2. Thus, this is a valid representation of $G_1 \sqcup G_2$ in D-space, as desired.

$\underline{\left[{\mu \atop M}\right]_k(G_1 \sqcup G_2) \geq D}$

It suffices to show that $\left[{\mu \atop M}\right]_k(G_1) \leq \left[{\mu \atop M}\right]_k(G_1 \sqcup G_2)$, as this would by symmetry imply that $\left[{\mu \atop M}\right]_k(G_2) \leq \left[{\mu \atop M}\right]_k(G_1 \sqcup G_2)$, and these give $\left[{\mu \atop M}\right]_k(G_1 \sqcup G_2) \geq D$.

Take a representation of $G_1 \sqcup G_2$ in $\left[{\mu \atop M}\right]_k(G_1 \sqcup G_2)$ dimensions. By removing all vertices of G_2, we are not creating any new edges (unobstructing potential visibility lines) in G_1 as by definition no edge exists between any pair of vertices $u \in G_1, v \in G_2$. This means that $\left[{\mu \atop M}\right]_k(G_1) \leq \left[{\mu \atop M}\right]_k(G_1 \sqcup G_2)$.

Thus, $\left[{\mu \atop M}\right]_k(G_1 \sqcup G_2) = D$, as desired. □

It easily follows that

Corollary 12. *Given graphs G_1 and G_2, the minimal embedding dimension of their disjoint union is*

$$\left[{\mu \atop M} \right]_k (G_1 \sqcup G_2) = \max\left(\left[{\mu \atop M} \right]_k (G_1), \left[{\mu \atop M} \right]_k (G_2), 2\right).$$

By repeatedly appylying Corollary 12, we obtain

Corollary 13. *Given two or more graphs $G_1, \ldots, G_m$,*

$$\left[{\mu \atop M} \right]_k (G_1 \sqcup \ldots \sqcup G_m) = \max\left(\left[{\mu \atop M} \right]_k (G_1), \ldots, \left[{\mu \atop M} \right]_k (G_m), 2\right).$$

From Theorem 6 and Corollary 13, we obtain:

Corollary 14. *Let $m \leq n$ be the size of the largest connected component of a graph G on n vertices. Then,*

$$\mu_k(G) \leq \max(2, m).$$

From Theorem 10 and Corollary 13, we obtain:

Corollary 15. *For a graph G on n vertices, where the largest connected component has $m \leq n$ vertices,*

$$M_k(G) \leq \max\left(2, \left\lceil \frac{m}{2} \right\rceil\right).$$

3.4.2 Cartesian Product

By looking at the projections onto smaller axis-parallel subspaces in general, we now show that the MED is subadditive under the Cartesian product [13] of multiple graphs.

Theorem 16. *The minimal embedding dimension of the Cartesian product of two graphs G_1 and G_2 as (U)RkVGs is bounded by*

$$\left[{\mu \atop M} \right]_k (G_1 \square G_2) \leq \left[{\mu \atop M} \right]_k (G_1) + \left[{\mu \atop M} \right]_k (G_2).$$

Proof. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be orthogonal $\left[{\mu \atop M} \right]_k (G_1)$ and $\left[{\mu \atop M} \right]_k (G_2)$ dimensional spaces in $\left[{\mu \atop M} \right]_k (G_1) + \left[{\mu \atop M} \right]_k (G_2)$ dimensions. Take representations of G_1 and G_2 in $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. For any two rectangles r_1 and r_2 in these respective representations, let R_{r_1,r_2} be the rectangle in $\left[{\mu \atop M} \right]_k (G_1) + \left[{\mu \atop M} \right]_k (G_2)$ dimensions of which r_1 and r_2 are projections. Note that there is an immediate bijection between $\{R_{r_1,r_2} \mid r_1 \in \mathcal{S}_1, r_2 \in \mathcal{S}_2\})$ and vertices in G_1 and G_2, namely, for any R_{r_1,r_2}, take the vertices corresponding to r_1 and r_2, respectively.

No two rectangles R_{s_1,s_2} and R_{t_1,t_2} overlap for any $s_1 \neq t_1, s_2 \neq t_2$, lest their projections would overlap in both of $\mathcal{S}_1$ and $\mathcal{S}_2$. If this were so, they would be the same in both projections, and therefore be the same rectangle.

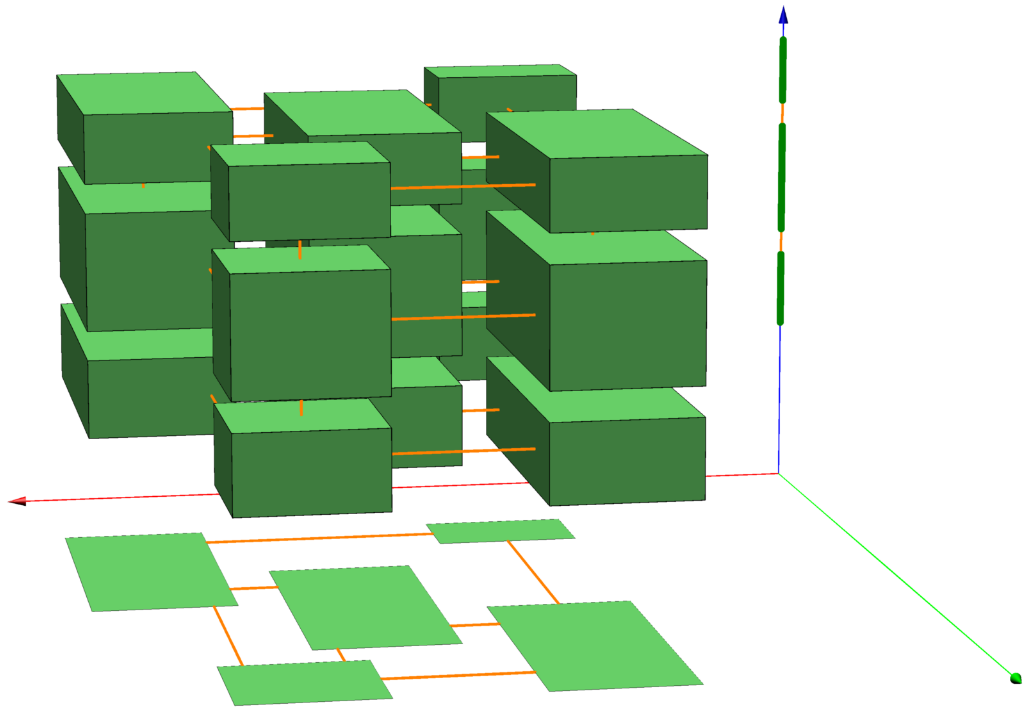


Figure 5: The Cartesian product of two graphs,
represented as the Cartesian product of their representations

Take the (U)RkVG of these rectangles. Given two adjacent rectangles, assume WLOG that the visibility line between these two rectangles is parallel to S_2; then in S_1 the projection of these two rectangles as well as any of the $\leq k$ rectangles that obstruct the visibility line overlap, and thus are the same projected rectangle. The projection of these two rectangles onto S_2 are adjacent, obstructed by the projections of the same $\leq k$ other rectangles. Conversely, if two rectangles in the projection onto S_1 are the same and in S_2 are adjacent or vice versa, the rectangles are adjacent.

Therefore, this is a valid representation of $G_1 \square G_2$ in $\left[{\mu \atop M}\right]_k(G_1) + \left[{\mu \atop M}\right]_k(G_2)$ dimensions, as desired. □

By repeatedly applying Theorem 16 on multiple graphs, we obtain:

Corollary 17. *The minimal embedding dimension of the Cartesian product of multiple graphs $G_1, \ldots, G_m$ as a (U)RkVG is bounded by*

$$\left[{\mu \atop M}\right]_k(G_1 \square G_2 \square G_3 \square \cdots \square G_m) \leq \left[{\mu \atop M}\right]_k(G_1) + \left[{\mu \atop M}\right]_k(G_2) + \left[{\mu \atop M}\right]_k(G_3) + \cdots + \left[{\mu \atop M}\right]_k(G_m).$$

3.4.3 Rooted Product

We'll now show a similar result for the rooted product [10].

Definition 18. Let G be a graph on n vertices, and let $\mathcal{H}$ be a sequence of n rooted graphs $H_1 \dots H_n$. The *rooted product* of G by $\mathcal{H}$, denoted $G(\mathcal{H})$, is the (unrooted) graph obtained by identifying the root of H_i with the i^{th} vertex of G for all $i \in [n]$.

Definition 19. Given a representation of a graph as an RkVG, and a side $\mathcal{S}$ of (open set with boundary) an axis-parallel $(d-1)$-dimensional hyperplane, the *expansion* of the representation by a distance L is formed by moving all the hyperrectangles' corners in $\mathcal{S}$ by a distance L orthogonally away from the hyperplane.

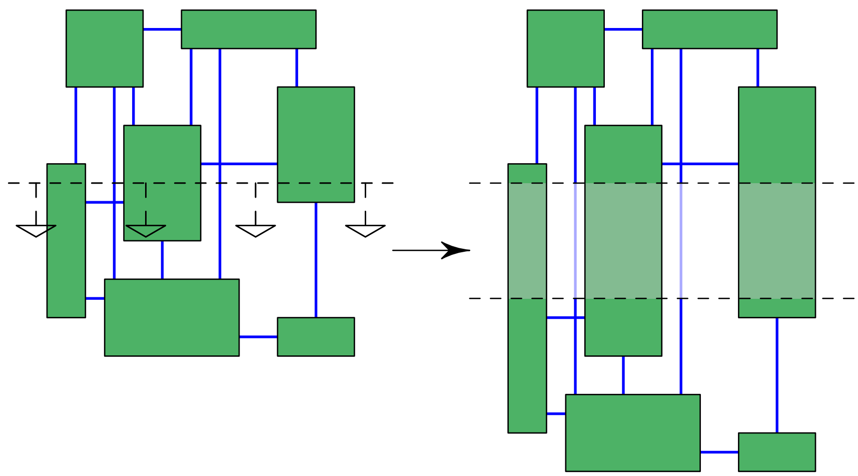


Figure 6: An expansion of an RVG representation

The expansion of a representation of a graph is another representation of the same graph, as all relationships are preserved.

Definition 20. Given a representation of a graph G as an RkVG and a vertex $v \in G$, the *inflation* of the representation at v by distance L is formed by expanding it on each side of a hyperplane containing a face of R not itself containing R, where R is the rectangle corresponding to v.

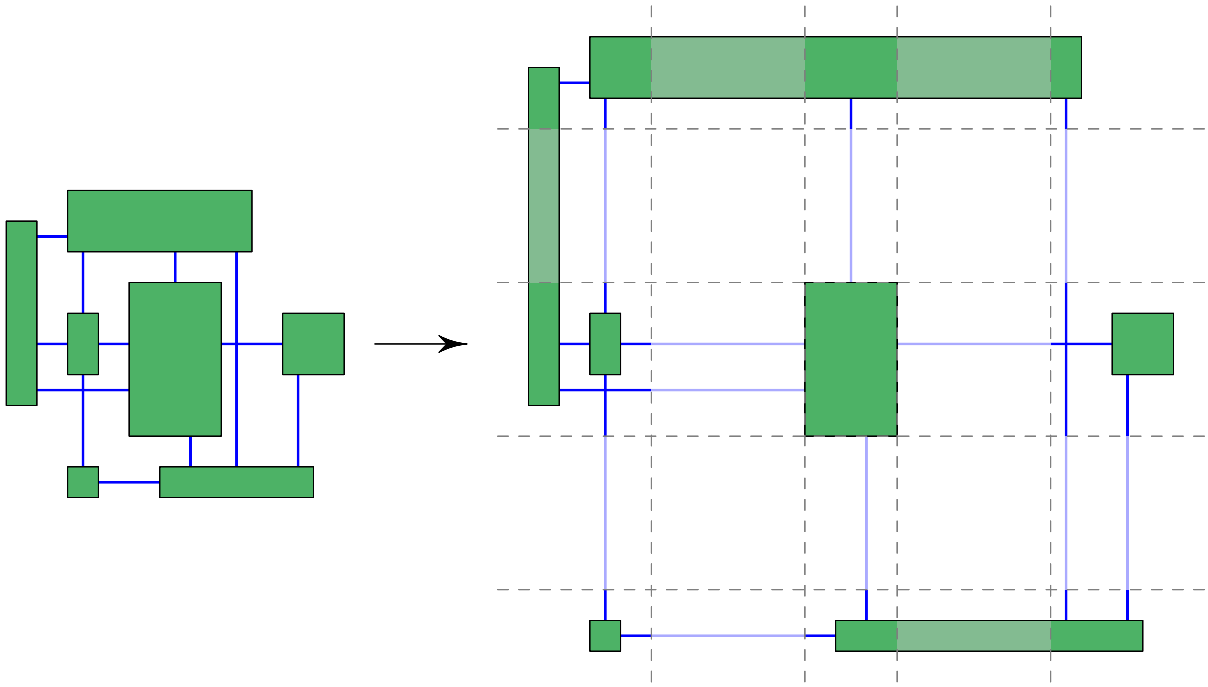


Figure 7: An inflation of an RVG representation

Theorem 21. *The minimal embedding dimension of the rooted product as a RkVG is bounded by*

$$\max\left(M_k(G), \max_{H\in\mathcal{H}}(M_k(H))\right) \leq M_k(G(\mathcal{H})) \leq M_k(G) + \max_{H\in\mathcal{H}}(M_k(H)).$$

Proof. For the lower bound, to establish that $M_k(G) \leq M_k(G(\mathcal{H}))$, we take any representation of $G(\mathcal{H})$ in $M_k(G(\mathcal{H}))$ dimensions. By definition, there's a naturally induced copy of G in $G(\mathcal{H})$.

Assume for the sake of contradiction that removing all vertices not in the induced G from $G(\mathcal{H})$ adds a visibility line to G. Consider the path formed by the vertices corresponding to the rectangles in $G(\mathcal{H})$ that this line intersects. This path must leave the induced G, a contradiction as it would self-intersect on its way back to G.

Then, to establish that $\forall H_i \in \mathcal{H},\ M_k(H_i) \leq M_k(G(\mathcal{H}))$, note that the natural copy of H_i in $G(\mathcal{H})$ is only connected to the rest of $G(\mathcal{H})$ at one vertex, so $G(\mathcal{H})$ is $H_i(\mathcal{G}_i)$ for some sequence of rooted graphs $\mathcal{G}_i$. Thus, $M_k(H_i) \leq M_k(H_i(\mathcal{G}_i)) = M_k(G(\mathcal{H}))$, as desired.

For the upper bound, by Lemma 9, we can take representations of $H_1, \ldots, H_n$ in $d = \max_{H\in\mathcal{H}}(M_k(H))$ dimensions. Rescale and translate all representations such that the rectangles corresponding to the roots are all unit size and centered at the origin. Let L be the smallest side length such that all representations fit inside an $\underbrace{L \times \cdots \times L}_{d}$ bounding rectangle.

As described in Definition 20, now inflate the representation of H_i around the root vertex by $(i-1) \times L$ for all $i \in [n]$, such that no rectangles beside the root vertex overlap between the representations.

Finally, take a representation of G in $M_k(G)$ dimensions. For all $i \in [n]$ and for $v \in H_i$, take the rectangle in $M_k(G) + \max_{H\in\mathcal{H}}(M_k(H))$ dimensions whose projection in the first $M_k(G)$ dimensions is the representation of the i^{th} vertex of G, and whose projection in the last $\max_{H\in\mathcal{H}}(M_k(H))$ dimensions is the representation of $v \in H_i$. We claim that these rectangles form a representation of $G(\mathcal{H})$.

Since all the roots of $\mathcal{H}$ have the same projection in the last $\max_{H\in\mathcal{H}}(M_k(H))$ dimensions, their visibilities are those of their projections in the first $M_k(G)$ dimensions; namely, the edges of G. Any rectangle that does not correspond to a root does not overlap with rectangles in the last $\max_{H\in\mathcal{H}}(M_k(H))$ by construction, and thus only sees those rectangles with which it overlaps in the first $M_k(G)$ dimensions and sees in the last $\max_{H\in\mathcal{H}}(M_k(H))$, as desired. □

3.4.4 Corona Product

We now look at the corona product, introduced by Frucht and Harary [9].

Definition 22. The *corona product* of two graphs G and H, denoted $G \odot H$, is obtained by taking one copy of G and $n = |V(G)|$ copies of H, and by connecting the i^{th} vertex of G to each vertex of the i^{th} copy of H for all $i \in [n]$.

Remark 23. For $\mathcal{H} = (H')_{i\in[n]}$ (i.e., H' repeated n times), where H' is H with an added universal root vertex (i.e., a root vertex connected to every other vertex of H), $G \odot H = G(\mathcal{H})$.

Theorem 24. *The minimal embedding dimension of the corona product of two graphs G and H as a RkVG is bounded by*

$$M_k(G) \leq M_k(G \odot H) \leq \max(M_k(G), M_k(H)) + 1.$$

Proof. By Remark 23 and Theorem 21, we have

$$M_k(G) \leq M_k(G(\mathcal{H})) = M_k(G \odot H),$$

where $\mathcal{H}$ is as in Remark 23.

We now show $M_k(G \odot H) \leq \max(M_k(G), M_k(H)) + 1$ by finding a $\max(M_k(G), M_k(H)) + 1$-dimensional representation of $G \odot H$.

By Lemma 9, we can take representations of G and H in $\max(M_k(G), M_k(H))$ dimensions. Shrink the representation of H until it is smaller than any one of the rectangles in the representation of G, and thicken both representations orthogonally by one unit into the d^{th} dimension, where $d = (\max(M_k(G), M_k(H)) + 1)$.

Take n copies of H's representation, corresponding to the n rectangles in the representation of G, and place them at different heights above the latter in the d^{th} dimension, such that each copy is exactly above its corresponding rectangle and no copies can see each other. As desired, any rectangle in G's representation now has a visibility line to every rectangle in exactly one copy of H's representation, with no visibility lines to or between other copies of H; moreover, visibilities are maintained within each of the original representations. □

4 Complete Graphs

We now construct an arrangement of rectangles where every rectangle can see every other rectangle, thus giving the complete graph.

4.1 Upper Bound

Lemma 25. *The complete graph on* $2(d-1)(k+1)+22(\lfloor k/2\rfloor+1)$ *vertices,* $K_{2(d-1)(k+1)+22(\lfloor k/2\rfloor+1)}$*, is a* d*-dimensional R*k*VG for* $d \geq 3$*.*

Proof. Figure 8, adapted from Figure 3 of [8], shows 22 rectangle projections, with one region in the center where all rectangles overlap in both their x and y coordinates. If for i from 0 to 21 we place a corresponding rectangle with thickness $0 < \epsilon < 1$ in 3-dimensional space at height $z = i$ above this plane, such that its projection to the plane is the rectangle labeled i, we thus obtain a 0-visibility representation of K_{22}, as in Figure 9.

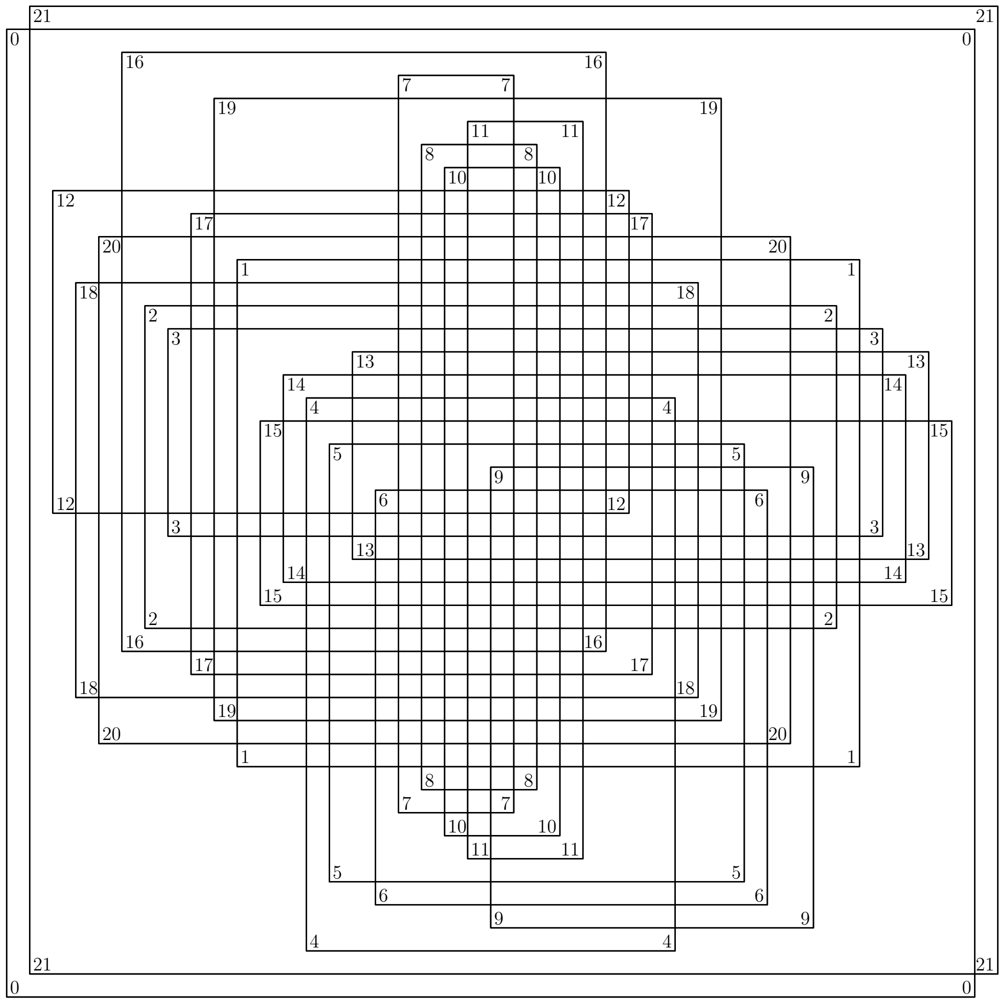


Figure 8: Projections of 22 rectangles, adapted from Figure 3 of [8]

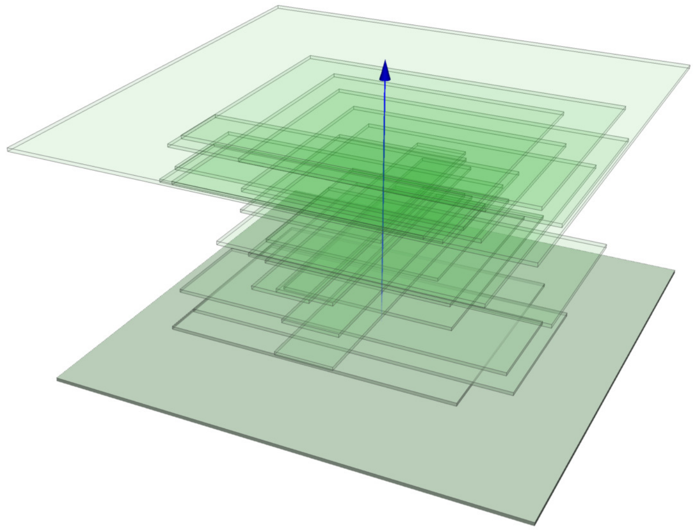


Figure 9: The 22 rectangles stacked on top of each other, seen from above the top right relative to Figure 8

If we place a rectangle of thickness $k = \frac{\epsilon}{\lfloor k/2 \rfloor + 1}$ for $0 < \epsilon < 1$ at height $i + \frac{j}{\lfloor k/2 \rfloor + 1}$ for all $i \in \{0, \ldots, 21\})$ and $j \in \left\{0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \right\})$, we now have a k-visibility representation of $K_{22(\lfloor k/2 \rfloor + 1)}$, where all rectangles are visible from any direction (top, bottom, and all four sides), as seen in Figure 10. (A visibility line between two rectangles with different projections passes throught at most $\lfloor k/2 \rfloor$ other rectangles with the same projection as each of the former and the latter rectangle).

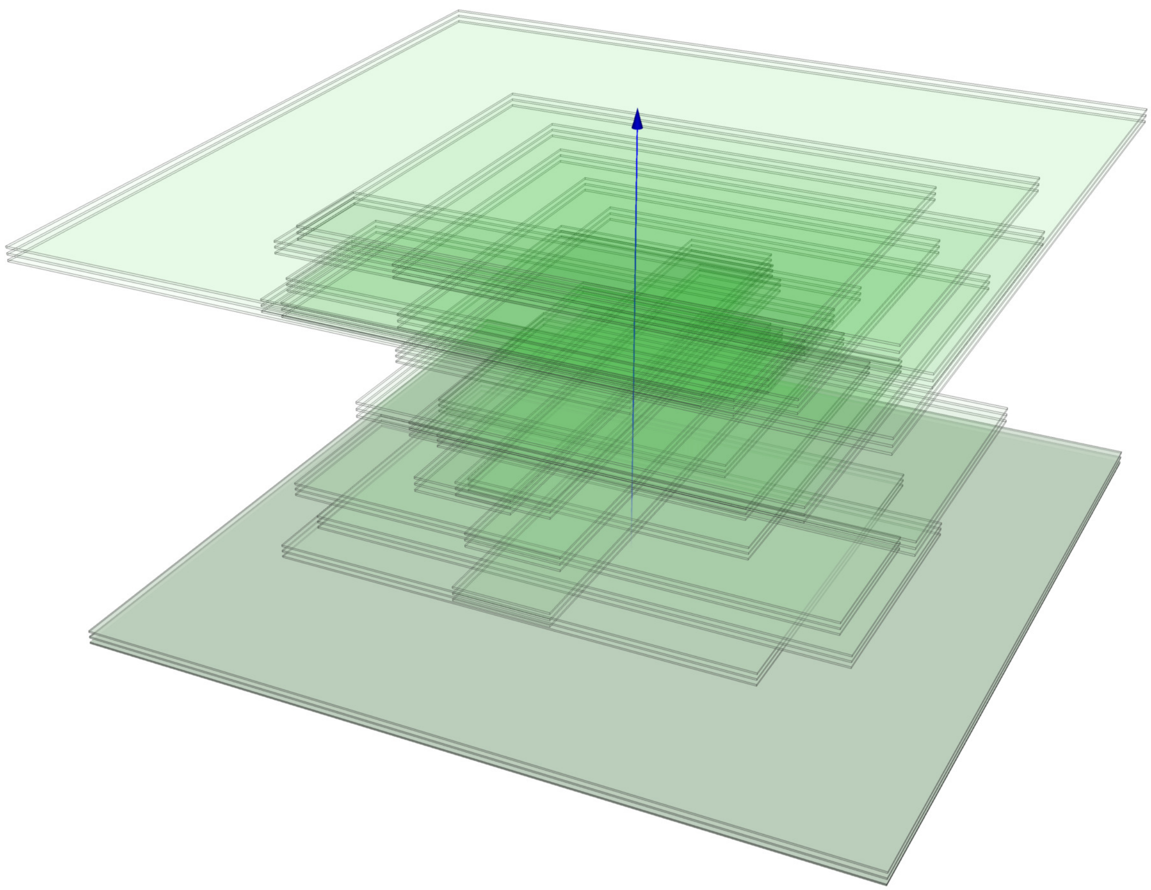


Figure 10: $22(\lfloor k/2 \rfloor + 1)$ rectangles stacked, representing $K_{22(\lfloor k/2 \rfloor+1)}$ with $k = 4$

We thicken this representation by one unit into each of the remaining $(d - 3)$ dimensions.

Finally, in each dimension except the 3rd (along whose axis in which we stack our $22(\lfloor k/2 \rfloor + 1)$ rectangles), we add $k + 1$ hyperrectangles in both directions from the center, at increasing distances and with increasingly large hyperfaces facing the center, such that each hyperrectangle has k-visibility to every other rectangle; i.e., such that the added rectangles in each dimension surround the entire representation up to that point. (See Figure 11 for an example).

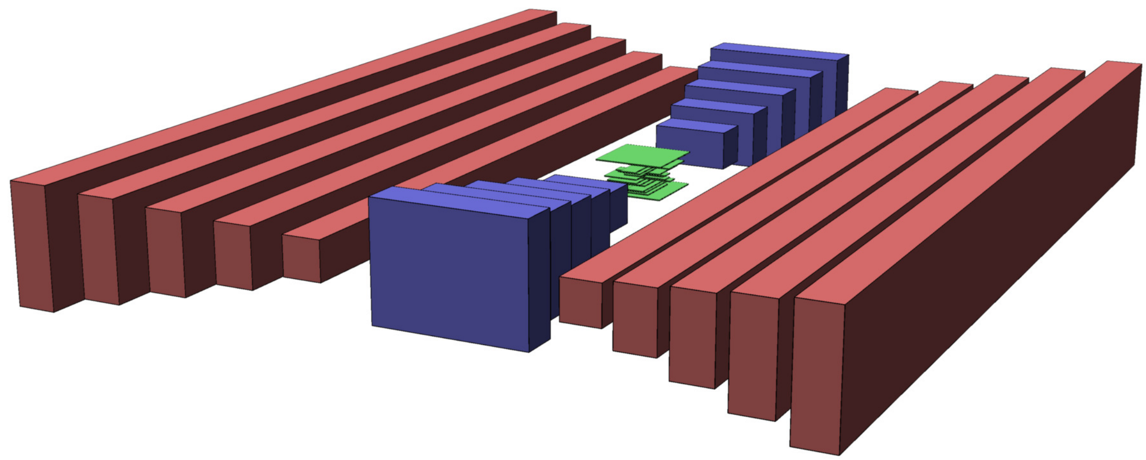


Figure 11: Representation of $K_{2(d-1)(k+1)+22(\lfloor k/2\rfloor+1)}$ with $k=4,\ d=3$

Along the i^{th} axis, there are $2(k+1)$ rectangles surrounding the center and the rectangles corresponding to prior axes, for a total of $2(d-1)(k+1)$ rectangles surrounding the center. As all rectangles, big and small, are k-visible to each other, we have a representation of $K_{2(d-1)(k+1)+22(\lfloor k/2\rfloor+1)}$. □

This gives a bound for the minimal embedding dimension of the complete graph:

Corollary 26. *The minimal embedding dimension of the complete graph on m vertices K_m as a RkVG is bounded by*

$$M_k(K_m) \leq \max\left(\left\lceil \frac{m-22(\lfloor k/2\rfloor+1)}{2(k+1)}\right\rceil + 1, 3\right).$$

4.2 Growth of $M_k(K_m)$

Lemma 27. *For some fixed k, let*

$$c_2 = 4k+5$$
$$c_i = \binom{c_{i-1}}{2} + 1 \ \mid\ i \geq 3.$$

Then, $K_{c_{2d-2}}$ cannot be represented in d dimensions with all visibility lines parallel.

To prove this lemma, we apply a technique used by Fekete et al. in Theorem 4 of [7].

Proof. We use induction on d.

Base case: $d=2$

Assume for the sake of contradiction that such a representation exists, and assume WLOG that all visibility lines are vertical. Flatten all rectangles so that they are horizontal line segments. We now have a bar k-visibility representation, as defined in the introduction, of $K_{c_2} = K_{4k+5}$. However as shown by Hartke et al. in *Further Results on Invisibility Graphs* [11], this is impossible, as desired.

Inductive step: $d - 1 \Rightarrow d$

We assume for the sake of contradiction that such a representation exists, and assume WLOG that all visibility lines are parallel to the first axis. As all rectangles overlap in every other dimension, there is then a line ℓ parallel to the d^{th} axis that passes through all c_{2d-2} rectangles.

Translate the coordinate system such that the origin lies on ℓ. Each rectangle has two faces orthogonal to the d^{th} axis, one on each side of ℓ. Let F_n and F'_n be the coordinates along the d^{th} axis of the corresponding faces for the n^{th} rectangle, where F_n is negative and F'_n is positive.

As shown by F.R.K Chung in *On unimodal subsequences* [4], every sequence of $\binom{a}{2}+1$ numbers has a subsequence of length a with one local maximum. Thus there exists a subsequence $(r_1, \ldots, r_{c_{2d-3}})$ among our c_{2d-2} rectangles such that the sequence $\left(-F_{r_1}, \ldots, -F_{r_{c_{2d-3}}}\right)$, has one local maximum.

Likewise, among these c_{2d-3} rectangles there is a sub-subsequence $(s_1, \ldots, s_{c_{2d-4}})$ such that the distance from ℓ to the second face of each orthogonal rectangle, $\left(F'_{s_1}, \ldots, F'_{s_{c_{2d-4}}}\right)$, form another unimaximal progression.

Note that in the d^{th} dimension, if rectangles s_i and s_k overlap for $i < j < k$, rectangle s_j contains their overlap. Thus, the visibility lines between these rectangles are those of their projections into the first $(d-1)$ dimensions. Thus, by induction, this is not possible.

□

Theorem 28. *The range of* $\left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k (K_m)$ *over* m *for fixed* k *is the set of nonnegative integers,* $\mathbb{N}_0$.

Proof. Let $r = R\left(\underbrace{c_{2d-2}, c_{2d-2}, \ldots, c_{2d-2}}_{d}\right)$ (adopting the notation from Lemma 27), where R denotes the multicolor Ramsey number function. Assume for the sake of contradiction that K_r is representable in d dimensions. Color each edge of K_r by the axis parallel to its visibility line. As this is a coloring with d colors of the edges of K_r, there is a monochromatic $K_{c_{2d-2}}$, contradicting Lemma 27. Thus, K_r is not representable in d dimensions.

Thus, no finite number of dimensions can represent K_m for all $m \in \mathbb{N}$, so $\left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k (K_m)$ takes on arbitrarily large values, and it suffices to show that $\left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k (K_{m+1}) \leq \left[\begin{smallmatrix}\mu\\M\end{smallmatrix}\right]_k (K_m) + 1$.

Assume that we have a representation of K_m in $\left[{\mu \atop M}\right]_k(K_m)$ dimensions. Add an extra dimension, thicken all the rectangles by 1 unit in this dimension, and replace one rectangle with two copies shifted by $-\frac{2}{3}$ and $\frac{2}{3}$ into the new dimension, respectively. Then, as all visibilities are maintained and the two copies can see each other, we have a representation of K_{m+1} in $\left[{\mu \atop M}\right]_k(K_m)+1$ dimensions, as desired. □

5 Complete Multipartite Graphs

To construct complete multipartite graphs, we arrange the rectangles in a "crosshatch".

Lemma 29. *The complete $(d-1)$-partite graph, $K_{m_1,\ldots,m_{d-1}}$, is a d-dimensional RkVG.*

Proof. Take an $m_1 \times m_2 \times \cdots \times m_{d-1}$ lattice in $(d-1)$-space. For every axis, take all axis-orthogonal $(d-2)$-spaces that pass through some of the lattice points. Cut all these spaces off to get axis-orthogonal $(d-2)$-dimensional hyperrectangles surrounding the lattice points. Add a small thickness to each of these spaces in their respective orthogonal dimensions.

For example, given $d=3, m_1=6, m_2=8$ we get the right hand side of Figure 12, and given $d=4, m_1=6, m_2=8, m_3=5$ we get the configuration in Figure 13.

Note that any pair of rectangles corresponding to spaces orthogonal to the same axis do not intersect, but rectangles corresponding to different axes do.

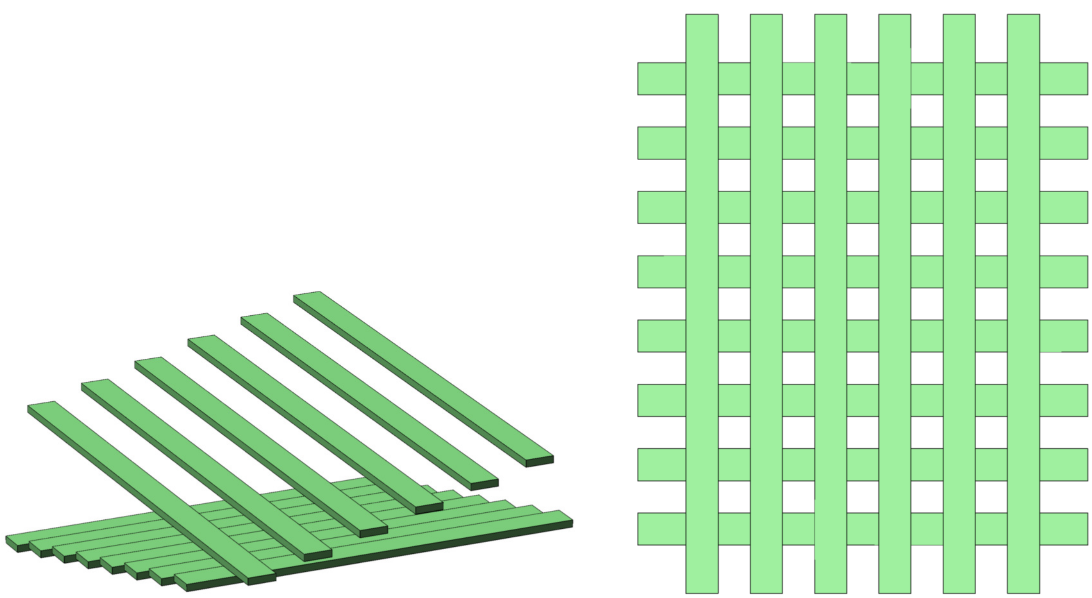


Figure 12: A representation of the 3-dimensional RVG $K_{6,8}$

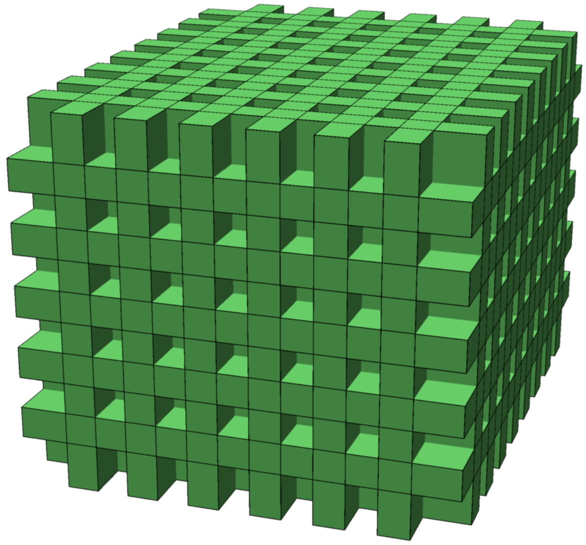


Figure 13: An overhead orthographic projection of a representation of the 4-dimensional RVG $K_{6,8,5}$

Now we extend the figure into the d^{th} dimension by adding a small thickness, and finally add a distinct height to each of them.

As any two rectangles corresponding to the same axis are not k-visible to each other, but any other two rectangles are, we have a representation of $K_{m_1,\ldots,m_{d-1}}$. □

This gives a bound for the minimal embedding dimension of the complete multipartite graph:

Corollary 30. *The minimal embedding dimension of the r-partite graph as a RkVG is bounded by*

$$M_k(K_{m_1,\ldots,m_r}) \leq r + 1.$$

6 Hypercubes

6.1 k-Visibility

Hypercubes are bipartite graphs. In the representation of (unit) rectangle k-visibility bipartite graphs for $k > 0$, we have to avoid rectangles seeing through other rectangles, as we would then create a triangle (e.g., rectangle A sees B sees C sees through B to A), which is not allowed in bipartite graphs.

Lemma 31. *The minimal embedding dimension of the hypercube graph on 2^m vertices, Q_m, as a (U)RkVG, respectively, is bounded by*

$$\left[{}^{\mu}_{M}\right]_k (Q_m) \leq \left\lceil \frac{2}{3} m \right\rceil .$$

Proof. Figure 14 shows a representation of Q_3 in 2 dimensions, so $M_k(Q_3) = \mu_k(Q_3) = 2$.

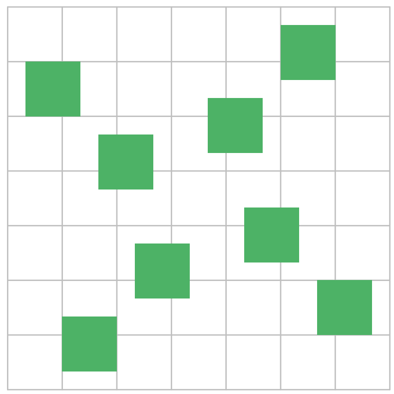


Figure 14: A (U)RkVG representation of the hypercube graph Q_3 in 2 dimensions

Since $\left[{}^{\mu}_{M}\right]_k (Q_1) = 1$, by Corollary 17 we get

$$\begin{aligned} \left[{}^{\mu}_{M}\right]_k (Q_m) &= \mu_k \left(\underbrace{Q_3 \square \cdots \square Q_3}_{\lfloor \frac{m}{3} \rfloor} \square \underbrace{Q_1 \square \cdots \square Q_1}_{m - 3\lfloor \frac{m}{3} \rfloor} \right) \\ &\leq \underbrace{\mu_k(Q_3) + \cdots \mu_k(Q_3)}_{\lfloor \frac{m}{3} \rfloor} + \underbrace{\mu_k(Q_1) + \cdots + \mu_k(Q_1)}_{m - 3\lfloor \frac{m}{3} \rfloor} \\ &= 2\left\lfloor \frac{m}{3} \right\rfloor + \left(m - 3\left\lfloor \frac{m}{3} \right\rfloor \right) \\ &= \left\lceil \frac{2}{3} m \right\rceil \end{aligned}$$

□

Remark 32. If a bipartite (U)RkVG has n vertices for $k > 0$, there can be at most $(n-1)$ vertical visibility lines in its representation, as its edges can form no triangles, and thus cycles. Similarly, there are at most $(n-1)$ horizontal visibility lines, and thus at most $2(n-1)$ total edges.

As Q_4 is bipartite, and has 16 vertices and $\frac{2^4 \cdot 4}{2} > 2(16-1)$ edges, it cannot be represented in $d = 2$ dimensions. Thus, Lemma 31 is tight for $m \leq 4, k > 0$.

6.2 0-Visibility

We now move on to 0-visibility, where as opposed to our previous construction, we do not have to worry about "collinear" rectangles.

Our 0-visibility representations of hypercubes can be placed in "grids". In the representation of Q_6 shown in Figure 15, we can see that the rectangles are organized in a $2^3 \times 2^3$ grid.

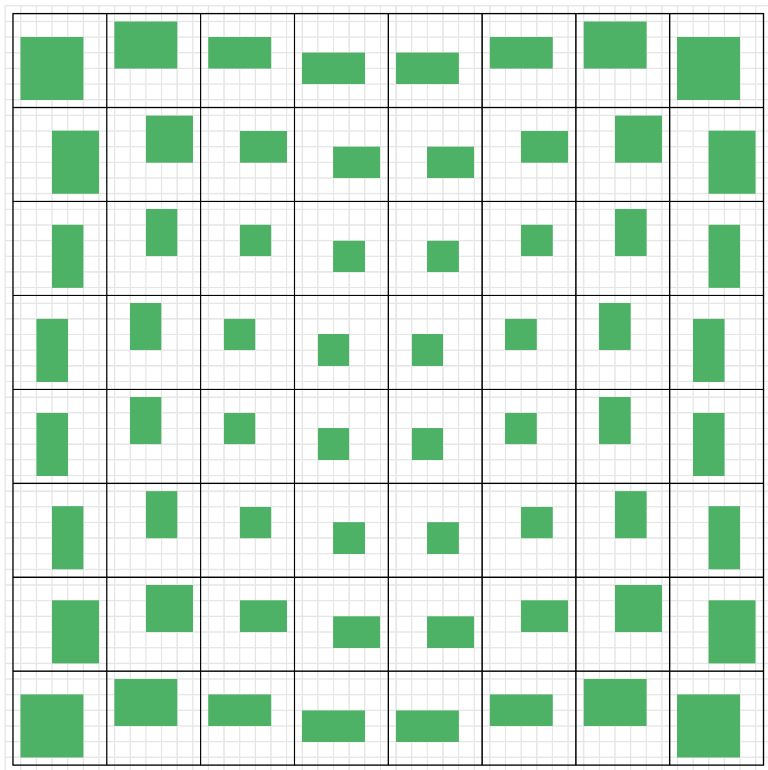


Figure 15: A 2-dimensional RVG representation of Q_6, the hypercube graph on 2^6 vertices, with a "grid-like structure"

In order to construct representations of hypercube graphs, we will first show how to construct the "columns", then show how to combine them into the full grid.

6.2.1 Gray Code

Before we proceed, we need to introduce the Gray code.

Definition 33. Gray code is a reordering of the binary numeral system such that two successive values differ in only one bit (binary digit) [14].

Like standard numbering systems (e.g., binary), Gray code representations of a number are implicitly padded with an infinite numbers of 0's on the left, and any number i is represented with a finite number of 1's. The number zero is represented with only 0's.

Given the Gray code representation of a non-negative integer i, the representation of $(i+1)$ is formed by flipping the least significant digit (i.e. the first one from the right) that introduces no redundancy with prior numbers.

In the following discussion, we will denote by $G_{i,j}$ digit $\#j$ of the Gray code representation of i, counting from the right such that $G_{i,0}$ is the least significant digit.

Example 34. The (four digit) Gray code representation of numbers 0 through 15 are shown in Figure 16.

0	0000	1000	15
1	0001	1001	14
2	0011	1011	13
3	0010	1010	12
4	0110	1110	11
5	0111	1111	10
6	0101	1101	9
7	0100	1100	8

Figure 16: The Grey code representations of numbers 0 through 15

We will make use of the following properties of Gray code:

- It is the "reflective binary code", where the representation of numbers $0, \ldots, (2^k - 1)$ are repeated in reverse order for numbers $2^k, \ldots, (2^{k+1} - 1)$, except that the k'th digit is 1 instead of 0 (with digit $\#0$ being the rightmost). In other words,

$$\forall\ i < 2^k \wedge j < k,\ G_{i,j} = G_{2^{k+1}-1-i,j}.$$

- $G_{2i,j+1} = G_{2i+1,j+1} = G_{i,j}$.
- $G_{i,0} = 0 \Leftrightarrow i \equiv \{0, 3\}) \pmod 4$.

6.2.2 Minimal Embedding Dimension as a URkVG

First we construct unit rectangle "columns":

Lemma 35. *The d-dimensional RVG formed by cubes of side length 2 centered at points of the form*

$$((d+2)i,\ G_{i,0},\ G_{i,1},\ G_{i,2},\ \ldots,\ G_{i,d-2})$$

for $0 \le i \le 2^d - 1$ is Q_d.

Proof. We use induction on d.

Base case: $d = 0$

This case trivially holds, as a single point is a valid representation of Q_0 in 0-dimensional space.

Inductive step: $d \Rightarrow d + 1$

For any rectangle centered at a point of the form

$$((d+2)i,\ G_{i,0},\ G_{i,1},\ \ldots,\ G_{i,d-2})$$

in d-space, we obtain two adjacent rectangles in $d + 1$ space with vectors

$$\begin{aligned} &((d+3)i,\ G_{i,0},\ G_{i,1}, \ldots,\ G_{i,d-2},\ G_{i,d-1}) \\ =&((d+3)i,\ G_{i,0},\ G_{i,1},\ \ldots,\ G_{i,d-2},\ 0) \end{aligned}$$

and

$$\begin{aligned} &\Big((d+3)\big((2^{d+1}-1)-i\big),\ G_{\left(2^{d+1}-1-i\right),0},\ G_{\left(2^{d+1}-1-i\right),1},\ \ldots,,\ G_{\left(2^{d+1}-1-i\right),d-1}\Big) \\ =&\big((d+3)\big((2^{d+1}-1)-i\big),\ G_{i,0},\ G_{i,1},\ \ldots,\ G_{i,d-2},\ 1\big), \end{aligned}$$

respectively.

By induction, all rectangles whose center coordinates end with 0 form a hypercube in d-space, likewise all rectangles whose center coordinates end with 1 form another hypercube in d-space. Since there can be no other visibility lines, they form a Q_{d+1} in $d + 1$-space.

This construction for $d = 3$ is shown in Figure 17.

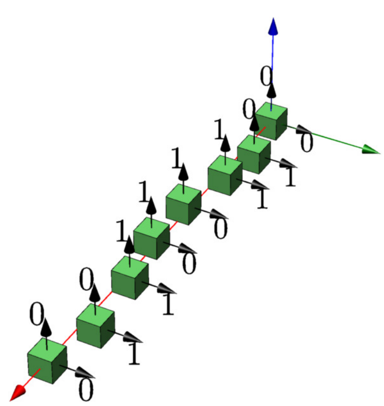


Figure 17: A 3-dimensional URVG representation of Q_3.
The x, y, and z axes are colored red, green, and blue, respectively.
The offset of the center of each cube from the x axis is indicated

□

Now we arrange such "columns" into a grid:

Theorem 36. *The hypercube graph on 2^{d^2} vertices, Q_{d^2}, is a d-dimensional URVG.*

Proof. For any tuple $(i_0, i_1, \ldots, i_{d-1})$ with $0 \leq i_j \leq 2^d - 1$ for all $j \in \{0, 1, \ldots, d-1\}$), take cubes of side length 2, where each cube is centered at the sum of the vectors

$$\begin{gathered}
\big((d+2)i_0,\ G_{i_0,0},\ G_{i_0,1},\ \ldots,\ G_{i_0,d-3},\ G_{i_0,d-2}\big),\\
\big(G_{i_1,d-2},\ (d+2)i_1,\ G_{i_1,0},\ \ldots,\ G_{i_1,d-4},\ G_{i_1,d-3}\big),\\
\big(G_{i_2,d-3},\ G_{i_2,d-2},\ (d+2)i_2,\ \ldots,\ G_{i_2,d-5},\ G_{i_2,d-4}\big),\\
\vdots\\
\big(G_{i_{d-2},1},\ G_{i_{d-2},2},\ G_{i_{d-2},3},\ \ldots,\ (d+2)i_{d-2},\ G_{i_{d-2},0}\big),\\
\big(G_{i_{d-1},0},\ G_{i_{d-1},1},\ G_{i_{d-1},2},\ \ldots,\ G_{i_{d-1},d-2},\ (d+2)i_{d-1}\big).
\end{gathered}$$

There are 2^{d^2} such tuples.

If we fix all but one of $i_0, i_1, \ldots, i_{d-1}$, the corresponding rectangles form a Q_d by Lemma 35. Since there can be no other visibility lines, they form a Q_{d^2}.

If we shrink this construction by a factor of 2, all the cubes become unit cubes. We see this construction applied to $d = 0$ through $d = 3$ in figures 18, 19, 20, and 21, respectively.

Figure 18: A 0-dimensional URVG representation of Q_0

Figure 19: A 1-dimensional URVG representation of Q_1

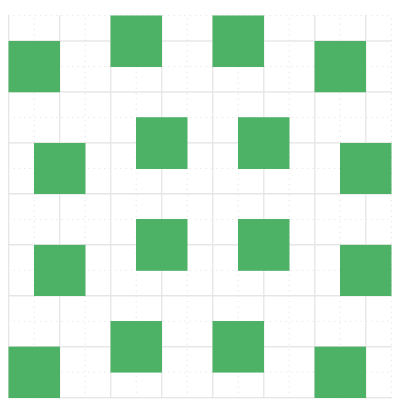


Figure 20: A 2-dimensional URVG representation of Q_4

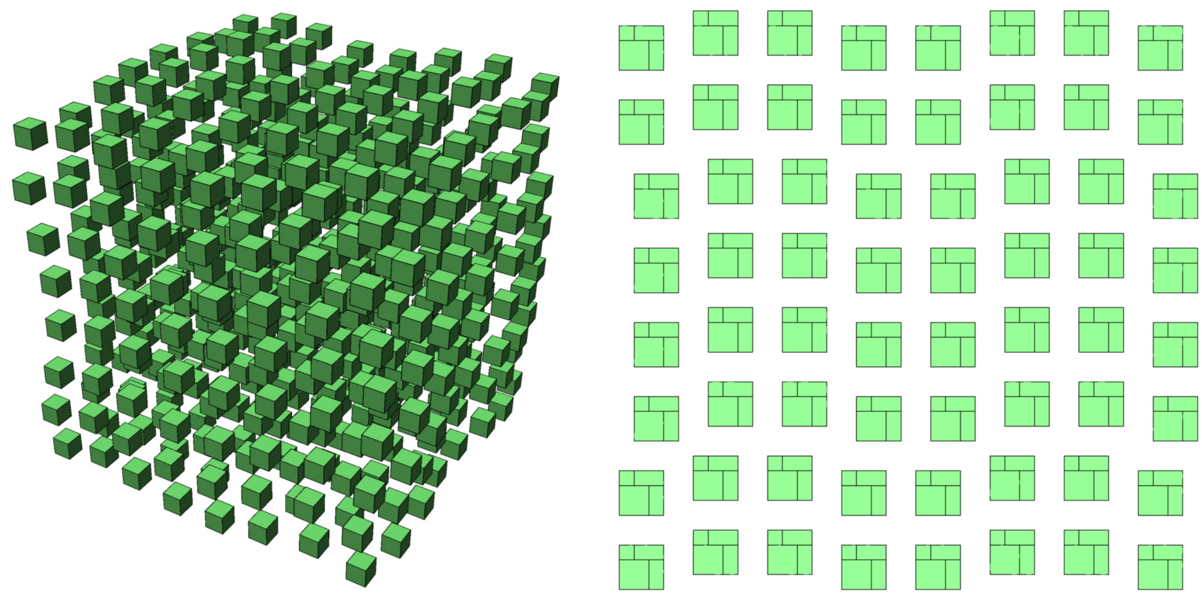


Figure 21: A 3-dimensional URVG representation of Q_9

□

We then obtain the following corollary:

Corollary 37. *The minimal embedding dimension of the m-dimensional hypercube graph as a URVG is bounded by*

$$\mu(Q_m) \leq \lceil \sqrt{m} \rceil.$$

6.2.3 Minimal Embedding Dimension as a RkVG

We proceed similarly for normal rectangles, again by first constructing rectangle "columns":

Lemma 38. *The d-dimensional RVG formed by rectangles with opposite vertices a_i and b_i, where $d \geq 2$, $0 \leq i < 2^{d+1} - 1$, and*

$$a_i = \begin{bmatrix} (d+4)i \\ -G_{i,d-2}G_{i,d-1} + 2G_{i,d-2} + G_{i,d-1} \\ G_{i,0} \\ G_{i,1} \\ \vdots \\ G_{i,d-3} \end{bmatrix}, \; b_i = \begin{bmatrix} (d+4)i + 4 \\ G_{i,d-2} - G_{i,d-1} + 4 \\ G_{i,0} + 4 \\ G_{i,1} + 4 \\ \vdots \\ G_{i,d-3} + 4 \end{bmatrix},$$

is Q_{d+1}.

Proof. We use induction on d.

<u>Base case: $d = 2$</u>

The resulting graph is shown in Figure 22.

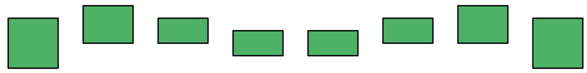


Figure 22: A 2-dimensional RVG representation of Q_3

<u>Inductive step: $d \Rightarrow d+1$</u>

Let $j_i = i + 2\left\lceil \frac{i}{2} \right\rceil, \quad k_i = 1 + i + 2\left\lfloor \frac{i}{2} \right\rfloor.$

Note that for all i,

- j_i and k_i are consecutive,
- the Gray code representation of j_i ends with the digit 0,
- the Gray code representation of k_i ends with 1,
- $G_{j_i,r} = G_{k_i,r} = G_{i,r-1}$ for all $r \in \mathbb{Z}^+$.

For the rectangle with opposite vertices a_i, b_i in d-space, we obtain two adjacent rectangles with respective opposite vertices at (c_{j_i}, d_{j_i}) and (c_{k_i}, d_{k_i}), where

$$c_{j_i} = \begin{bmatrix} (d+5)j_i \\ -G_{j_i,d-1}G_{j_i,d} + 2G_{j_i,d-1} + G_{j_i,d} \\ G_{j_i,0} \\ G_{j_i,1} \\ G_{j_i,2} \\ \vdots \\ G_{j_i,d-2} \end{bmatrix} = \begin{bmatrix} (d+5)j_i \\ -G_{i,d-2}G_{i,d-1} + 2G_{i,d-2} + G_{i,d-1} \\ 0 \\ G_{i,0} \\ G_{i,1} \\ \vdots \\ G_{i,d-3} \end{bmatrix},$$

$$d_{j_i} = \begin{bmatrix} (d+5)j_i + 4 \\ G_{j_i,d-1} - G_{j_i,d} + 4 \\ G_{j_i,0} + 4 \\ G_{j_i,1} + 4 \\ G_{j_i,2} + 4 \\ \vdots \\ G_{j_i,d-2} + 4 \end{bmatrix} = \begin{bmatrix} (d+5)j_i + 4 \\ G_{i,d-2} - G_{i,d-1} + 4 \\ 4 \\ G_{i,0} + 4 \\ G_{i,1} + 4 \\ \vdots \\ G_{i,d-3} + 4 \end{bmatrix}$$

and

$$c_{k_i} = \begin{bmatrix} (d+5)k_i \\ -G_{k_i,d-1}G_{k_i,d} + 2G_{k_i,d-1} + G_{k_i,d} \\ G_{k_i,0} \\ G_{k_i,1} \\ G_{k_i,2} \\ \vdots \\ G_{k_i,d-2} \end{bmatrix} = \begin{bmatrix} (d+5)k_i \\ -G_{i,d-2}G_{i,d-1} + 2G_{i,d-2} + G_{i,d-1} \\ 1 \\ G_{i,0} \\ G_{i,1} \\ \vdots \\ G_{i,d-3} \end{bmatrix},$$

$$d_{k_i} = \begin{bmatrix} (d+5)k_i + 4 \\ G_{k_i,d-1} - G_{k_i,d} + 4 \\ G_{k_i,0} + 4 \\ G_{k_i,1} + 4 \\ G_{k_i,2} + 4 \\ \vdots \\ G_{k_i,d-2} + 4 \end{bmatrix} = \begin{bmatrix} (d+5)k_i + 4 \\ G_{i,d-2} - G_{i,d-1} + 4 \\ 5 \\ G_{i,0} + 4 \\ G_{i,1} + 4 \\ \vdots \\ G_{i,d-3} + 4 \end{bmatrix},$$

respectively.

By the inductive hypothesis, all rectangles of the first type as well as all rectangles of the second type form a Q_{d+1} in d-space. Since there can be no other visibility lines, they form a Q_{d+2} in $d+1$-space.

□

Again we arrange such "columns" into a grid:

Theorem 39. *The hypercube graph Q_{d^2+d} is a d-dimensional RVG for $d \geq 2$.*

Proof. For any tuple $(i_0, i_1, \ldots, i_{d-1})$ with $0 \leq i_j \leq 2^{d+1} - 1$ for all $j \in \{0, 1, \ldots, d-1\}$), take a rectangle with opposite corners at

$$\sum_{j=0}^{d-1} \begin{bmatrix} 0 & & & & & & & 1 \\ 1 & 0 & & & & \text{\Huge 0} & & \\ & 1 & 0 & & & & & \\ & & 1 & \ddots & & & & \\ & & & \ddots & \ddots & & & \\ & & & & 1 & 0 & & \\ & \text{\Huge 0} & & & & 1 & 0 & \\ & & & & & & 1 & 0 \end{bmatrix}^j \cdot \begin{bmatrix} (d+4)i \\ -G_{i,d-2}G_{i,d-1} + 2G_{i,d-2} + G_{i,d-1} \\ G_{i,0} \\ G_{i,1} \\ \vdots \\ G_{i,d-3} \end{bmatrix}$$

and

$$\sum_{j=0}^{d-1} \begin{bmatrix} 0 & & & & & & & 1 \\ 1 & 0 & & & \mathbf{0} & & & \\ & 1 & 0 & & & & & \\ & & 1 & \ddots & & & & \\ & & & \ddots & \ddots & & & \\ & & & & 1 & 0 & & \\ & \mathbf{0} & & & & 1 & 0 & \\ & & & & & & 1 & 0 \end{bmatrix}^j \cdot \begin{bmatrix} (d+4)i+4 \\ G_{i,d-2} - G_{i,d-1} + 4 \\ G_{i,0} + 4 \\ G_{i,1} + 4 \\ \vdots \\ G_{i,d-3} + 4 \end{bmatrix}.$$

There are 2^{d^2+d} such tuples.

If we fix all but one of $i_0, i_1, \ldots, i_d$, the corresponding rectangles form a Q_{d+1} by Lemma 35. Since there can be no other visibility lines, they form a $Q_{d(d+1)} = Q_{d^2+d}$.

This construction is applied to $d = 2$ and $d = 3$ in figures 23 and 24, respectively.

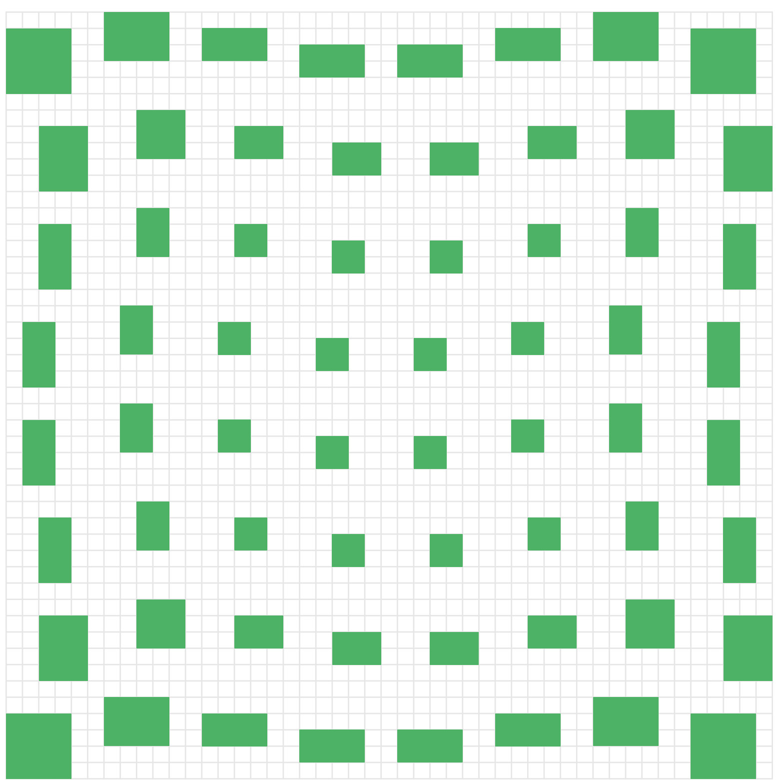


Figure 23: A 2-dimensional RVG representation of Q_6

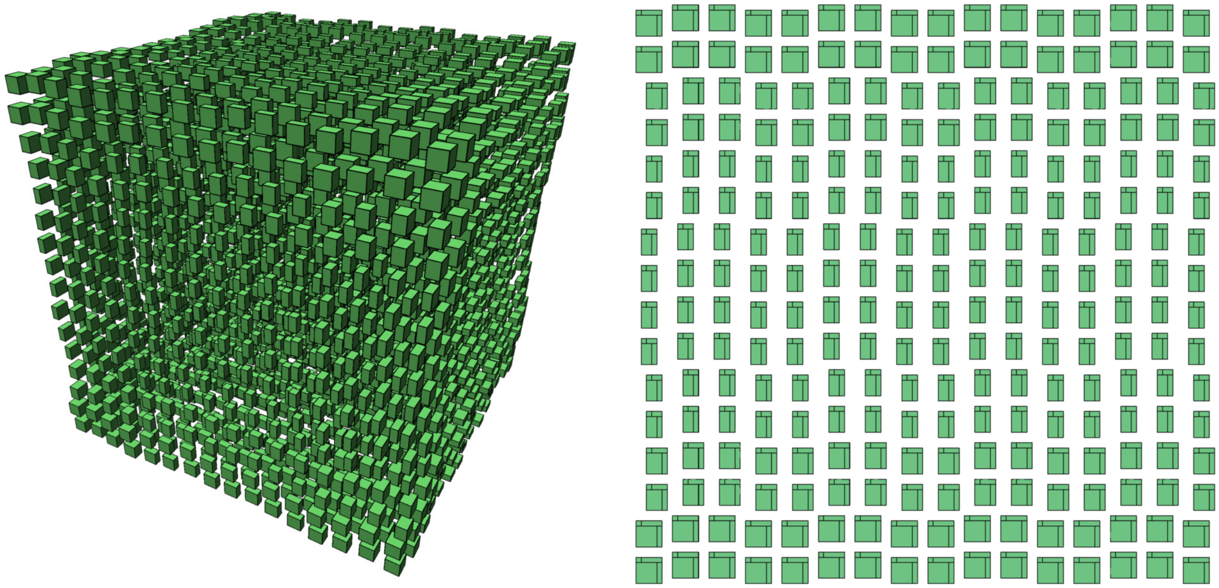


Figure 24: A 3-dimensional RVG representation of Q_{12}

$\square$

This gives

Corollary 40. *For all $m \neq 2$, the minimal embedding dimension of the m-dimensional hypercube graph as a RVG is bounded by*

$$M(Q_m) \leq \lfloor \sqrt{m} \rceil$$

(where $\lfloor x \rceil$ denotes x rounded to the nearest integer).

Remark 41. The minimal embedding dimension d of hypercube graphs Q_m as RVGs include the following:

- $M(Q_0) = 0$. A representation of Q_0 in 0 dimensions is shown in Figure 18.
- $M(Q_1) = 1$. A representation of Q_1 in 1 dimension is shown in Figure 19. Q_1 cannot be represented in 0 dimensions, since there is no room for two "rectangles".
- $M(Q_i) = 2 \mid i \in \{2, \ldots, 6\})$. A Q_6 in 2-space is shown in Figure 23. Q_5, Q_4, Q_3 and Q_2 can be obtained by repeatedly removing the right or top half of the boxes in these configurations, thus ending up with $2^5, 2^4, 2^3$ and 2^2 rectangles in each respective representation.

  None of these graphs can be represented in 1-space, where the only thing that can be represented is paths. Thus, the minimal embedding dimension of Q_2 through Q_6 is $d = 2$.
- $M(Q_i) = 3 \mid i \in \{8, \ldots, 12\})$. A representation of Q_{12} in 3-space is shown in Figure 24. There are $16^3 = 4096$ boxes in this representation, corresponding to $n = 2^{12} = 4096$ vertices in Q_{12}.

$Q_{11},\ Q_{10},\ Q_9$ and Q_8 can be obtained by repeatedly removing the top half of the boxes in these configurations, thus ending up with $n = 2^{11}, 2^{10}, 2^9$, and 2^8 boxes in each respective representation.

Given $m \geq 8$, the number of edges in Q_m is $\frac{m}{2} \cdot 2^m \geq 4 \cdot 2^m = 4n$.

Dean and Hutchinson found that a bipartite 2-dimensional RVG on $n \geq 4$ vertices has at most $4n - 12$ edges [5].

Since Q_m is bipartite, and $4n > 4n - 12$, it follows that $Q_{m\geq 8}$ is not representable in 2 dimensions. Thus, the minimal embedding dimensions of Q_8 through Q_{12} are all $d = 3$.

7 Acknowledgments

I am grateful for my wonderful mentor, Dr. Jesse Geneson of Iowa State University, who inspired me to delve into the topic of visibility graphs in the first place and has continued to guide me throughout this project. I would also like to thank Dr. Pavel Etingof, Dr. Slava Gerovitch, and Dr. Tanya Khovanova of the MIT PRIMES-USA program, and the Mathematics Department of the Massachusetts Institute of Technology for the opportunity to work on this research project.

7.1 Biography

Mentee: Espen Slettnes

Espen is a student research intern in the MIT PRIMES-USA mentorship program, where he has been receiving guidance from Dr. Jesse Geneson of Iowa State University for the past two years. Espen and Dr. Geneson have previously co-authored two research papers in graph theory.

Having attended the UC Berkeley affiliated Berkeley Math Circle since age six, Espen started teaching college-level math topics in the Advanced group for avid math students in grades 11-12 at age thirteen. During his middle school years, he took six upper-division math courses through the UC Berkeley Extension Concurrent Enrollment program for pre-college students, maintaining a GPA of 4.0. Now in 9th grade at Stanford Online High School, he is simultaneously taking graduate-level math and undergraduate physics courses at Berkeley.

Espen received state and national recognitions for his research projects in the United States, including the Project of the Year Award at the 2018 California Science and Engineering Fair, First Prize in Mathematics at the 2018 Broadcom MASTERS, and an Outstanding Student Poster Presentation Award at the 2019 Joint Mathematics Meetings. He qualified for the USA Mathematical Olympiad in 2019, won a bronze medal in the USA Physics Olympiad 2018, and has been competing in the USA Computing Olympiad since 2016. Chosen twice to receive a Spirit of Ramanujan Fellowship in 2018 and 2019, Espen used the scholarly development prizes to attend the Canada/USA Mathcamp and AwesomeMath summer programs where he took short courses in a variety of math topics at undergraduate to graduate level.

Mentor: Dr. Jesse Geneson

Jesse Geneson graduated from Harvard University in 2010, received a Ph.D. in applied mathematics from the Massachusetts Institute of Technology in 2015, and is currently a postdoctoral scholar at Iowa State University. He has mentored MIT PRIMES projects and research projects at MIT including the winner of the 2014 Siemens individual competition. Results of these projects were published in the Electronic Journal of Combinatorics, Journal of Graph Algorithms and Applications, and Discrete Mathematics.

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参赛队员 / Team Member(s)：

Espen Slettnes

指导老师 / Mentor:

Jesse Geneson, Ph.D.

September 1, 2019

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