COVER PAGE: OPTIMAL SOLUTIONS AND RANKS IN THE MAX-CUT SDP

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OPTIMAL SOLUTIONS AND RANKS
IN THE MAX-CUT SDP

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ABSTRACT. The max-cut problem is a classical graph theory problem
which is NP-complete. The best polynomial time approximation scheme
relies on *semidefinite programming* (SDP). We study the conditions under
which graphs of certain classes have rank 1 solutions to the max-cut SDP.
We apply these findings to look at how solutions to the max-cut SDP
behave under simple combinatorial constructions. Our results determine
when solutions to the max-cut SDP for cycle graphs is rank 1. We find
the solutions to the max-cut SDP of the vertex sum of two graphs. We
then characterize the SDP solutions upon joining two triangle graphs by
an edge sum.

Keywords: Max-cut, Semidefinite program, Rank, Cycle graphs, Vertex sum, Edge sum

CONTENTS

1. INTRODUCTION

Consider a graph $G = (V, E)$ with vertex set $V = [n]$ and fixed weights $\{w_{ij}\}_{ij \in E}$ assigned to the edges. Consider partitions, or *cuts,* of the vertex set $V = V_1 \sqcup V_2$. The *max-cut problem* asks for the maximum possible sum of all weights between vertices on opposite sides of the cut. In other words, we wish to maximize

$$\text{cut}(V_1, V_2) := \sum_{i \in V_1, j \in V_2} w_{ij}$$

across all partitions $V = V_1 \sqcup V_2$.

The max-cut problem is a key problem in theoretical computer science and operations research. In particular, it is one of Karp's 21 NP-complete problems [10]. Furthermore, it has applications in physics and circuit design [3]. In statistical physics and magnetism, the Edwards-Anderson model for the ground states of spin glasses is an optimization problem in ± 1-variables and can be reduced to the max-cut problem. In Very-Large-Integrated-Scale (VLSI) circuit design, the problem of reducing the number of vias, or connections between layers of a circuit board, can be reduced to the max-cut problem.

Although the max-cut problem is NP-complete [10], a breakthrough paper of Goemans and Williamson [9] proved that the max-cut problem can be approximated in polynomial time up to a factor of 0.87854 using a technique known as *semidefinite programming* (SDP).

A semidefinite program is an optimization problem where we minimize a linear function on the entries of a matrix subject to two types of constraints: the matrix is *positive semidefinite* (i.e., its eigenvalues are nonnegative) and its entries satisfy some linear equations.

Semidefinite programs have far-reaching applications [16, 17], most notably in approximation algorithms, which we discuss in this paper.

The max-cut problem can be relaxed to a semidefinite program. The max-cut SDP requires the following definitions:

The *Laplacian matrix* of a weighted graph $L(G, w)$ is the $n \times n$ symmetric matrix with entries:

$$L(G, w)_{ij} := \begin{cases} -w_{ij} & \text{if } ij \in E \\ \sum_k w_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

We use the shorthand $X \succeq 0$ to indicate that the matrix X is positive semidefinite, and we use $\bullet$ to represent the Frobenius inner product: $A \bullet B = \text{Tr}(A^T B)$ for square matrices A and B of the same dimension. Furthermore, $\mathbb{S}^n$ denotes the space of $n \times n$ symmetric matrices.

The max-cut SDP is presented below, the derivation of which we will present in Section 2:

$$\max_{X \in \mathbb{S}^n} \quad \frac{1}{4} L(G, w) \bullet X$$
$$\text{s.t.} \quad X_{ii} = 1 \text{ for } i \in [n]$$
$$X \succeq 0.$$

In this paper, we study how the optimal solution to the max-cut SDP behaves under simple graph operations. The operations covered in the paper are the *vertex sum*, where two graphs are joined at a common vertex, and the *edge sum*, where two graphs are joined at a common edge. These operations are special instances of a *clique sum*, where two graphs are joined at a common clique K_n for some n.

In this paper we play special attention to the rank of the optimal solution obtained. In general, low rank solutions to the max-cut SDP are desirable. In particular, if the optimal solution is rank 1 then we may recover an exact optimal solution of the max-cut problem. Our current results characterize when cycle graphs have rank 1 solutions to their max-cut SDPs. We also study the behaviour for random weights. For random normally distributed weights, we experimentally estimate the probability that the optimal solution is rank 1.

Related work. The max-cut SDP was first studied in [8], where the authors showed promising experimental results. It gained major attention after the seminal work of Goemans and Williamson [9], where they introduced a rounding technique capable of producing a cut which is optimal up to a factor of 0.87854. The gives the best known polynomial time approximation algorithm to the max-cut problem.

Having a low rank solution has many important computational and theoretical consequences. Indeed, if the optimal solution has low rank then one can take advantage of faster optimization routines [5], and the approximation factor can be also be improved [2, 7].

The rank of the optimal solution matrix is closely related to the geometric structure of the SDP. The geometry of the max-cut SDP has been investigated in [6, 12, 13]. The paper [14] leverages geometric information to provide upper bounds on the rank of the solution.

It is possible to construct more sophisticated SDPs for the max-cut problem that rely on a technique known as the *SOS method* [4]. It is an open problem whether such sophisticated SDPs can improve the approximation factor. This problem has received quite a lot of interest in recent years, and it is closely related to a twenty-year old conjecture from computer science known as the *unique games conjecture* [11]. We do not consider these advanced SDPs here.

Structure. The structure of this paper is as follows. In Section 2, we derive and state the max-cut semidefinite program and provide definitions related to weighted graphs and clique sums of graphs. In Section 3, we discuss

experimental findings on the ranks of solutions to the max-cut SDP. In Section 4, we discuss theorems which characterize the ranks of solutions to the max-cut SDP, as well as solutions to the max-cut SDP for vertex sums and an edge sum of triangles. In Section 5, we provide proofs for theorems discussed in Section 4. In Section 6, we discuss unresolved conjectures not covered in Section 5 and potential future avenues for research.

2. Background

2.1. **Semidefinite programs.** Let $C \in \mathbb{S}^n$ be an $n \times n$ cost matrix. Consider m constraint matrices $A_1, A_2, \ldots, A_m \in \mathbb{S}^n$, as well as a constraint vector $b \in \mathbb{R}^m$. A *semidefinite program* is an optimization problem of the form

$$
\begin{aligned}
\max_{X} \quad & C \bullet X \\
\text{s.t.} \quad & A_i \bullet X = b_i \ \forall 1 \leq i \leq m \\
& X \succeq 0
\end{aligned}
$$

For each semidefinite program, there exists an associated *dual semidefinite program*, formulated as follows, where b, A_i and C are the same as above:

$$
\begin{aligned}
\min_{y,S} \quad & b^T y \\
\text{s.t.} \quad & S = \sum_{i=1}^{n} y_i A_i - C \\
& S \succeq 0
\end{aligned}
$$

It is always the case that the primal value p^* is at most the dual value d^*, or $p^* \leq d^*$. This is known as *weak duality*. In case that the two values agree, or $p^* = d^*$, we say that *strong duality* holds. It is known that strong duality holds under mild assumptions, see e.g. [16, Thm 3.1].

2.2. **Max-cut SDP.** In this section we explain the connection between the max-cut problem and its SDP relaxation.

Given a cut (V_1, V_2), consider assigning a 1 to all vertices in V_1 and a -1 to all vertices in V_2. Then we can treat a cut as a vector x with $x_i^2 = 1$ for all i. Now, the quantity $1 - x_i x_j$, for any edge $ij \in E$, will be 0 if i and j are in the same group, and 2 otherwise. Thus, we arrive at

$$
\text{cut}(V_1, V_2) = \sum_{i \in V_1, j \in V_2} w_{ij} = \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - x_i x_j) = \frac{1}{4} \sum_{i,j} L(G, w)_{ij} x_i x_j.
$$

Let $X = xx^T \in \mathbb{S}^n$. Note that X is rank 1, positive semidefinite, and has all diagonal entries equal to 1. Moreover, all such matrices X satisfying those three conditions can be rewritten as $X = xx^T$ for some vector x with $x_i^2 = 1$ for all i.

Thus, the max-cut problem can be rephrased as

$$\max_{V_1,V_2} \; \text{cut}(V_1,V_2) = \begin{array}{l} \max\limits_{x} \; \frac{1}{4}x^T L(G,w)x \\ \text{s.t} \quad x_i^2 = 1 \;\; \forall i \end{array} = \begin{array}{l} \max\limits_{X} \; \frac{1}{4}L(G,w) \bullet X \\ \text{s.t} \quad X_{ii} = 1 \;\; \forall i \\ \qquad X \succeq 0, \; \text{rank}\, X = 1 \end{array}$$

The last optimization problem involves the nonconvex constraint $\text{rank}\, X = 1$. We can relax this problem to a semidefinite program by getting rid of the rank 1 constraint. By doing so, we arrive at the primal max-cut SDP.

Definition 1. Let $C = \frac{1}{4}L(G,w)$. The *primal* max-cut SDP is the following relaxation of the max-cut problem:

$$\max_{X \in \mathbb{S}^n} \quad C \bullet X$$
$$\text{s.t.} \quad X_{ii} = 1 \text{ for } i \in [n]$$
$$X \succeq 0.$$

Henceforth, the max-cut SDP will refer to the primal max-cut SDP.

Note that if the optimal solution X of the SDP has rank 1, then we may write it in the form $X = xx^T$, and hence we may recover the optimal cut. We say that the relaxation is *exact* if this happens.

We now derive the dual of this semidefinite program. Note that in the situation above, the constraint variables A_i and b_i correspond to each diagonal entry on the matrix. In particular, A_i is a matrix with iith entry 1 and all other entries 0, while $b_i = 1$. We can then define the dual for the max-cut SDP.

Definition 2. The *dual* max-cut SDP is as follows:

$$\min_{y \in \mathbb{R}^n, S \in \mathbb{S}^n} \quad \sum y_i$$
$$\text{s.t.} \quad S = \text{Diag}(y) - C$$
$$S \succeq 0.$$

The following is a well known theorem that characterizes the optimal solutions to the max-cut SDP.

Theorem 1 ([16, eq.(33)]). *Let matrices $\bar{X}, \bar{S}$ satisfy the following conditions:*

- *$\bar{X}$ is primal feasible, or is a valid solution to the primal SDP (not necessarily optimal). In other words, $\bar{X}$ satisfies $A_i \bullet \bar{X} = b_i$ for all $1 \le i \le m$ and $\bar{X} \succeq 0$.*
- *$\bar{S}$ is dual feasible, or is a valid solution to the dual SDP (not necessarily optimal). In other words, $\bar{S} = \sum_{i=1}^m y_i A_i - C$ for some value of y, and $\bar{S} \succeq 0$.*
- *Complementary slackness, or $\bar{X}\bar{S} = 0$.*

Then, $\bar{X}$ and $\bar{S}$ are optimal solutions to the primal and dual SDP, respectively, and strong duality holds.

2.3. **Clique sums of graphs.** In this paper, we study ranks of solutions to the primal max-cut SDP in relation to clique-sums. A *clique-sum* of two graphs G and H which both contain a clique graph K_n, and we join G and H along this clique graph K_n to form a new graph F, which has two vertex sets V_1 and V_2 such that the following are true:

- The induced subgraph of F on V_1 is isomorphic to G,
- The induced subgraph of F on V_2 is isomorphic to H,
- The induced subgraph of F on $V_1 \cap V_2$ is isomorphic to K_n, and
- $V_1 \cup V_2$ is the vertex set of H.

We then define a *vertex sum* as a clique-sum with the clique K_1, and an *edge sum* as a clique-sum with the clique K_2.

3. Experiments on rank of max-cut SDP

Rank is important to the max-cut problem because lower rank solutions will be able to yield better approximate solutions to the max-cut SDP. We have already noted that a rank 1 solution will exactly find the solution to the max-cut problem, while it has been shown that rank 2 solutions yield better approximation algorithms for the max-cut problem.

Given a weighted graph G, the set of optimal primal (or dual) solutions might not be unique. We let $r_P(G)$ be the largest rank among all possible primal optimal solutions. Similarly, we may define $r_D(G)$ to be the largest rank among all possible dual optimal solutions. We say that strict complementarity holds if $r_P(G) + r_D(G) = n$. It is known that strict complementarity holds generically [1]. Hence, we restrict our attention to the primal rank $r_P(G)$.

We study the primal ranks $r_P(G)$ for random families of weighted graphs. For a given graph $G = ([n], E)$, we consider two probability distributions on the weights $\{w_{ij} : ij \in E\}$. In the first probability distribution, each weight w_{ij} is an independent Gaussian random variable. The second probability distribution is similar, except that we force the weights w_{ij} to be nonnegative.

The probabilities shown in Table 1 are the experimental values for the probability that the solution to the primal max-cut SDP of a graph returns a certain rank.

To obtain this data, we generated 1000 different weights for each graph $G = (V, E)$ corresponding to random points on the $|E|$-dimensional unit sphere (for arbitrary weights), then 1000 different weights corresponding to random points on the $|E|$-dimensional unit sphere in the first orthant (for positive weights). We then counted the number of times that each of the ranks was obtained.

Known algorithms will return the highest rank solution, so graphs which yield multiple solutions or solutions of multiple different ranks will not be shown in the following table.

In Table 1, "Diamond" represents a K_4 graph with the edge removed, "Butterfly" is the vertex sum of K_3 and K_3, and the "Fish" graph is the vertex sum of K_3 and C_4.

TABLE 1. Rank distribution of optimal solutions

	Arbitrary Weights			Positive Weights		
Graph	Rank 1	Rank 2	Rank 3	Rank 1	Rank 2	Rank 3
K_3	85%	15%	0%	69%	31%	0%
C_4	77%	23%	0%	100%	0%	0%
Diamond	71%	29%	0%	65%	35%	0%
C_5	73%	27%	0%	45%	55%	0%
Butterfly	72%	25%	3%	50%	42%	8%
C_6	70%	30%	0%	100%	0%	0%
Fish	62%	34%	4%	69%	31%	0%

We know a few facts about the ranks of the optimal matrices. In particular, there always exists a primal matrix $\bar{X}$ with rank r such that $\binom{r+1}{2} \leq n$, and there always exists a dual matrix $\bar{S}$ with rank s such that $s \leq n-1$, see [15, Thm 2.1].

However, there can also exist solutions with rank greater than p. For instance, the butterfly graph has an approximately 2% chance of returning a rank 3 primal solution, despite the fact that there must exist an optimal solution of rank 2 or lower. When this occurs, we know that the primal solution is not unique. For most graphs, there will be a unique solution to the primal matrix.

4. Solutions and Ranks for Particular Classes of Graphs

4.1. **Rank 1 solutions for cycles.** We aim to understand the conditions in which the rank of a primal solution is 1 for certain graphs.

We will prove the following theorem in Section 5, regarding the rank of solutions to the primal max-cut SDPs cycle graphs in the following section:

Theorem 2. *Consider a cycle graph C_n with $V = [n]$ and E consisting of all edges $(i, i+1)$ for $1 \leq i \leq n-1$, as well as $(n, 1)$. For simplicity, let w_i denote the weight of $(i, i+1)$ for $1 \leq i \leq n-1$, and w_n be the weight of $(n, 1)$. There exists a rank 1 solution to the primal SDP $L(G, w)$ if and only if at least one of the following statements is true:*

- *There are an even number of positively weighted edges in the graph.*
- *There exists a weight w_m such that*

$$\frac{1}{|w_m|} \geq \sum_{i \neq m} \frac{1}{|w_i|}.$$

Corollary 1. *If a cycle graph has a rank 1 solution to its corresponding max-cut primal SDP, then the rank 1 solution is unique.*

As an application of the above theorem, we calculate the probability that the K_3 graph has a rank 1 primal solution in the following example.

Example 1. The probability that a a triangle, or K_3, with weights randomly chosen from a standard distribution, has a rank 1 primal solution, is $\frac{6-2\sqrt{3}}{3}$. When restricted to random positive weights, this probability is $\frac{9-4\sqrt{3}}{3}$.

Suppose we randomly choose weights w_1, w_2, w_3 in $\mathbb{R}^3$. Without loss of generality, we can normalize this vector so that $w_1^2 + w_2^2 + w_3^2 = 1$. We claim that the condition of Theorem 2 is satisfied if and only if either $w_1 + w_2 + w_3, w_1 - w_2 - w_3, -w_1 + w_2 - w_3$, or $-w_1 - w_2 + w_3$ is at most -1. Indeed, note that if there are an even number of positively weighted edges, one of these 4 values has 3 negative terms, and thus as $|x| > x^2$ for all $x^2 \leq 1$, it satisfies this condition. Otherwise, note that

$$\frac{1}{|w_1|} \geq \frac{1}{|w_2|} + \frac{1}{|w_3|} \iff |w_2 w_3| \geq |w_1 w_2| + |w_1 w_3|$$

$$\iff (|w_1| - |w_2| - |w_3|)^2 \geq 1$$

Looking at each octant individually, we see that the resulting signs will define the four inequalities as described. Using calculus, it can be shown the probability that one of the four inequalities is satisfied can be shown to be $p = \frac{1}{2} - \frac{\sqrt{3}}{6}$. As a result, the probability the solution to the primal max-cut SDP of the triangle with random weights is rank 1 is $4p = \dfrac{6 - 2\sqrt{3}}{3} \approx 0.8453$. Since $0 < \operatorname{rank} \bar{X} \leq 3 - \operatorname{rank} \bar{S} \leq 2$, then the remaining probability is the probability the primal is rank 2.

Note that when we randomly choose weights, we have a $\frac{1}{2}$ chance of having an even number of positive weights, thus this total probability is the average of 1 and the probability for an odd number of positive weights. As a result, when we restrict our random weights to be all positive, the probability is

$$2\left(\frac{6 - 2\sqrt{3}}{3}\right) - 1 = \frac{9 - 4\sqrt{3}}{3} \approx 0.6905.$$

4.2. **Max-cut SDP for a vertex sum.** We have characterized the conditions where cycle graphs are rank 1. We now characterize the probability distributions of primal solutions graphs that are the vertex sum of two graphs, in terms of the distributions of the subgraphs.

Say p_1 and p_2 be the probabilities that the rank of the solution to the primal max-cut SDP for K_3 with arbitrary weights is 1 and 2, respectively.

According to the experimental values in the table, the probability for arbitrary weights that the rank of the primal matrix is 1 is p_1^2, the probability it is rank 2 is $2p_1 p_2$, and the probability it is rank 3 is p_2^2.

We obtain similar results for the Fish graph. Furthermore, equivalent results hold for the normally distributed nonnegative weights. These observations regarding the probability distributions motivates the following theorem:

Suppose we have two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$, where $L(G_1)$ has primal-dual solution pair X_1, S_1, and $L(G_2)$ has primal-dual solution pair X_2, S_2.

Without loss of generality suppose we sum the graphs along the last element of V_1 and the first of V_2. Let x_1 be the final column of X_1, and as the final entry of x_1 is 1, define y_1 so that $x_1 = \begin{bmatrix} y_1 \\ 1 \end{bmatrix}$. Similarly, let $x_2 = \begin{bmatrix} 1 \\ y_2 \end{bmatrix}$ be the first column of X_2. Finally, define Y_1, Y_2 so that

$$X_1 = \begin{bmatrix} Y_1 & y_1 \\ y_1^T & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 & y_2^T \\ y_2 & Y_2 \end{bmatrix}.$$

Theorem 3. *The primal-dual pair*

$$X' = \begin{bmatrix} Y_1 & y_1 & y_1 y_2^T \\ y_1^T & 1 & y_2^T \\ y_2 y_1^T & y_2 & Y_2 \end{bmatrix}, S' = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S_2 \end{bmatrix}$$

is optimal. Furthermore, $\operatorname{rank} X' = \operatorname{rank} X_1 + \operatorname{rank} X_2 - 1$.

A direct consequence of this result is that we now have well-defined solutions to the primal and dual max-cut SDPs of two graphs summed at a vertex. The following corollary shows that in fact the max-cut problem for a vertex sum graph also has well-defined ranks for its solutions.

Corollary 2. *If G' is a vertex sum of G_1 and G_2 along an element of V_1 and an element of V_2, then there exists a primal solution to $L(G')$ with rank at most* $\max(\operatorname{rank}(X_1), \operatorname{rank}(X_2))$.

The proof of the above theorem and corollary is presented in Section 5. To illustrate how this theorem helps us further determine the probabilities, we provide the following example:

Example 2. The probabilities for the butterfly graph primal SDP returning a solution of rank 1, 2, and 3 are in fact $p_1^2, 2p_1 p_2$, and p_2^2, respectively, where $p_1 = \dfrac{6 - 2\sqrt{3}}{3}$ and $p_2 = 1 - p_1 = \dfrac{2\sqrt{3} - 3}{3}$.

Note that the best known algorithms return the highest rank solutions. As a result, a rank 1 solution will be returned if and only if the two summed K_3 graphs are both yield rank 1 primal solutions, a rank 3 solution if and only if both yield rank 2 solutions, and a rank 2 solution otherwise. This gives us the probabilities desired.

4.3. Max-cut SDP for a diamond graph.

In addition to our study of vertex sums, we also studied edge sums. In particular, we focused on the diamond graph, which is the edge sum of two K_3 graphs.

Suppose that we have a *diamond* graph with $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$. Let X^*, S^* be a primal-dual solution pair to the max-cut SDP for G. Let G_1 and G_2 be the induced subgraphs of

G on vertex sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$ respectively with same weight vector as G.
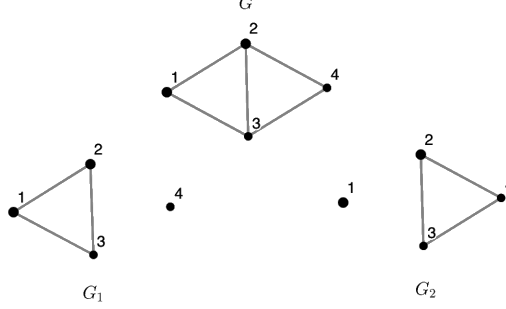
We prove the following two theorems about butterfly graphs.

Theorem 4. *Assume that G_1 and G_2 both yield rank 1 optimal primal solutions which agree on the joined edge between vertices 2 and 3. Then the following are equivalent:*

- $w_{23} \leq \min(\frac{1}{1/w_{12}+1/w_{13}}, \frac{1}{1/w_{24}+1/w_{34}})$
- *The matrix X, where $x = [-1, 1, 1, -1]$ and $X = xx^T$, is an optimal primal solution.*

Also, the above items imply that the optimal dual solution S^* is of the form

$$S^* = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S_2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -w_1 & w_1 & 0 \\ 0 & w_1 & -w_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 5. *Assume that G_1 and G_2 both yield rank 1 optimal primal solutions which agree on the joined edge between vertices 2 and 3. Then the following three are equivalent:*

- $w_{23} \geq \frac{1}{|1/w_{12}-1/w_{13}|} + \frac{1}{|1/w_{24}-1/w_{34}|}$
- *The matrix X, where $x = [\epsilon_1, -1, 1, \epsilon_2]$ and $X = xx^T$, is an optimal primal solution for some $\epsilon_1, \epsilon_2 \in \{-1, 1\}$.*

The above items imply that the optimal dual solution S^* is of the form

$$S^* = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S_2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & w_1 & w_1 & 0 \\ 0 & w_1 & w_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. Proofs

5.1. **General Lemma on Rank 1 optimal solutions.** We first prove the following result, which will be very helpful in the proofs in the following sections:

From Theorem 1, we can obtain that if x is the ones vector, then $\bar{S}x = 0 \iff \bar{S} = -\frac{1}{4}L$. Therefore, the ones matrix is an optimal solution to the

primal max-cut SDP if and only if and only if $\bar{S} \succeq 0$ when $\bar{S}x = 0$, which is equivalent to $-L(G, w) \succeq 0$.

We can generalize to the following lemma, for any vector $x \in \pm 1^n$:

Lemma 1. *Suppose a graph G has weight vector w and the primal SDP of the matrix $L(G, w)$ has a rank 1 optimal solution. There there exists a vector x with $x_i^2 = 1$ for all i such that the matrix S' defined as follows*

$$S' = L(G, \overline{w})$$
$$\overline{w_{ij}} = -x_i x_j w_{ij} \ \forall ij \in E(G)$$

is positive semi-definite.

Proof. Suppose the primal SDP returns a solution $(\bar{X}, \bar{S})$ with $\bar{X}$ rank 1. By definition, we know $\bar{X} = xx^T$ for some vector x with $x_i = \pm 1$ for all i. Since x is the first column of X, then $Sx = 0$. Define S' with $S'_{ij} = 4x_i x_j \bar{S}_{ij}$. Then for all edges $i \in E(G)j$, we just have $S'_{ij} = x_i x_j \cdot 4S_{ij} = -x_i x_j w_{ij}$. It is evident that $S'_{ij} = 0$ for ij not connected. Let $\bar{S}_i$ be the ith column of $\bar{S}$, and note by definition $x^T S_i = 0$. Now,

$$\sum_{j=1}^{n} S'_{ij} = \sum_{j=1}^{n} 4x_i x_j \bar{S}_{ij} = 4x_i(x^T \bar{S}_i) = 0.$$

Thus, S' is indeed the Laplacian matrix with the weights as claimed. Finally, note that for any vector y, we have

$$y^T S' y = \sum_{1 \le i,j \le n} 4y_i y_j x_i x_j \bar{S}_{ij}.$$

Setting a vector z with $z_i = x_i y_i$, we see $\bar{S} \succeq 0 \implies z^T \bar{S}z \ge 0$ and thus $y^T S' y \ge 0$. As a result, S' is positive semidefinite. $\qquad\square$

This proof tells us if $\bar{X}$ is rank 1 and is an optimal solution to the primal max-cut SDP for some graph, then some transformation of L is positive semi-definite. More practically, the set of constraints on L to have a rank 1 optimal primal solution is orthogonal to the condition for which the all ones matrix is an optimal primal solution.

5.2. **Rank 1 solutions for cycles.**

Proof of Theorem 2. By Lemma 1, we know that a rank 1 solution will lead to the exact optimal cut x with $x_i^2 = 1$ for all i. For convenience let $x_{n+1} = x_1$. Suppose that $w_k x_k x_{k+1}$ is positive for at least two different values of k, say $k_1 < k_2$. Then consider the vector x' with $x'_k = -x_k$ for all $k_1 < k \le k_2$ and $x'_k = x_k$ otherwise. It is clear that the total sum of $w_k x_k x_{k+1}$ will decrease, thus leading to a better primal solution. As a result, $w_i x_i x_{i+1}$ is positive for at most one value of i.

If $-w_i x_i x_{i+1} \geq 0$ for all values of i, then we can take the product of all of these constraints to get

$$\prod_{i=1}^{n} -w_i = \prod_{i=1}^{n} -w_i x_i x_{i+1} \geq 0.$$

Thus there are an even number of positive w_i in this case. In fact, the converse is true: if there are an even number of positive w_i, it is always possible to choose an x so that $w_i x_i x_{i+1}$ is negative for all i.

Now, without loss of generality suppose that x is the ones vector, so that the matrix $S' = -L(G, w)$ is positive semidefinite by Lemma 1. $S' \succeq 0$ if and only if the determinant of each principal submatrix is positive. Note that S' has the form

$$\begin{bmatrix} -(w_n + w_1) & w_1 & \dots & 0 & w_n \\ w_1 & w_1 + w_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -(w_{n-2} + w_{n-1}) & w_{n-1} \\ w_n & 0 & \dots & w_{n-1} & -(w_{n-1} + w_n) \end{bmatrix}.$$

By induction, we can show that the determinant of the lower-right $m \times m$ submatrix of S' is the mth symmetric sum of the numbers $-w_{n-m}, -w_{n-m+1}, \dots, -w_n$ for $1 \leq m < n$. Call this symmetric sum d_m, and note that for $mge2$,

$$d_m = -w_{n-m} d_{m-1} + (-1)^m w_{n-m+1} \dots w_{n-1} w_n.$$

The base cases of $m = 1$ and $m = 2$ are easily verified, giving determinants of d_1 and d_2. Assuming that the hypothesis is true for m and $m - 1$ with $m \geq 2$, we can evaluate the determinant of the lower-right $m + 1 \times m + 1$ matrix to get

$$\begin{aligned} & -(w_{n-m-1} + w_{n-m}) d_m + w_{n-m}^2 d_{m-1} \\ = & -w_{n-m-1} d_m - w_{n-m}(d_m - w_{n-m} d_{m-1}) \\ = & -w_{n-m-1} d_m - w_{n-m}(-1)^m w_{n-m+1} \dots w_{n-1} w_n \\ = & \quad d_{m+1}. \end{aligned}$$

We only need to consider determinants of principal submatrices that correspond to connected components of the cycle graph, because other can be expressed in terms of those. Note that if all w_i are negative, we automatically satisfy all the necessary inequalities. Otherwise, we have at least one constraint $d_{n-1} \geq 0$ and note that for w_i all nonzero it is equivalent to

$$(-w_1)(-w_2) \dots (-w_n) \left(\frac{-1}{w_1} + \dots + \frac{-1}{w_n} \right) \geq 0.$$

As shown earlier, at most one w_i can be positive, so if exactly one w_i is positive, then we must have

$$\frac{1}{w_1} + \dots + \frac{1}{w_n} \geq 0,$$

as desired. All other constraints follow similarly from this single constraint; thus, this constraint is both necessary and sufficient to describe when the cycle graph has a solution with x as the all ones vector, where not all weights are negative. For any other vector x, the weights w_i will simply be replaced instead by the values $x_i x_{i+1} w_i$, and and the final necessary and sufficient constraint is that there exists a weight w_m such that

$$\frac{1}{|w_m|} \geq \sum_{i \neq m} \frac{1}{|w_i|}.$$

$\square$

Proof of Corollary 1. If the first condition (an even number of positive edges) is satisfied, then the vector x is the unique vector (up to multiplication by -1) so that every $w_i x_i x_{i+1}$ is negative.

Otherwise, we have some weight w_m such that

$$\frac{1}{|w_m|} \geq \sum_{i \neq m} \frac{1}{|w_i|}.$$

Note that clearly, $|w_m| < |w_i|$ for all $i \neq m$, and as a result m is unique. The the unique solution to the max-cut problem (and thus the unique solution to the max-cut SDP) is the vector x such that $w_i x_i x_{i+1}$ is negative for all $i \neq m$, and positive for $i = m$.

In both cases, a rank 1 solution for the cycle graph must be unique. $\square$

5.3. **Max-cut SDP for a vertex sum.**

Proof of Theorem 3. Due to the dual feasibility of S_1, S_2 we know S' is a sum of two positive semidefinite matrices, so evidently S' is dual feasible. Now note $X' = \begin{bmatrix} X_1 & x_1 y_2^T \\ y_2 x_1^T & Y_2 \end{bmatrix} = \begin{bmatrix} Y_1 & y_1 x_2^T \\ x_2 y_1^T & X_2 \end{bmatrix}$. From here, we see

$$X'S' = \begin{bmatrix} X_1 & x_1 y_2^T \\ y_2 x_1^T & Y_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Y_1 & y_1 x_2^T \\ x_2 y_1^T & X_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & S_2 \end{bmatrix}$$

$$= \begin{bmatrix} X_1 S_1 & 0 \\ y_2 x_1^T S_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & y_1 x_2^T S_2 \\ 0 & X_2 S_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From the fact that $X_1 S_1 = X_2 S_2 = 0$.

Now, let V be a matrix with n columns and column size equal to the rank of X_1 such that $V^T V = X_1$. Let v be the final column of v. Then we know $V^T v$ is the last column of X_1, which is x_1, and similarly $v^T v = 1$. Note that

$$\begin{bmatrix} V^T \\ y_2 v^T \end{bmatrix} \begin{bmatrix} V & v y_2^T \end{bmatrix} = \begin{bmatrix} V^T V & y_2 v^T V \\ V^T v y_2^T & y_2 v^T v y_2^T \end{bmatrix} = \begin{bmatrix} X_1 & y_2 x_1^T \\ x_1 y_2^T & y_2 y_2^T \end{bmatrix}.$$

Furthermore, note that by Schur Complement, $X_2 \succeq 0, 1 \succ 0 \implies Y_2 \succeq y_2 y_2^T$. As a result,

$$X' = \begin{bmatrix} V & v y_2^T \end{bmatrix}^T \begin{bmatrix} V & v y_2^T \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Y_2 - y_2 y_2^T \end{bmatrix} \succeq 0.$$

Finally, Let V_2 be a matrix with $V_2^T V_2 = Y_2 - y_2 y_2^T$. Then note

$$X' = \begin{bmatrix} V & vy_2^T \\ 0 & V_2 \end{bmatrix}^T \begin{bmatrix} V & vy_2^T \\ 0 & V_2 \end{bmatrix}^T.$$

Since such a representation is an upper block diagonal matrix, we know $\operatorname{rank} X' = \operatorname{rank} V + \operatorname{rank} V_2 = \operatorname{rank}(V^T V) + \operatorname{rank}(V_2^T V_2) = \operatorname{rank}(X_1) + \operatorname{rank}(Y_2 - y_2 y_2^T)$. By properties of Schur Complement, we thus have $\operatorname{rank} X' = \operatorname{rank} X_1 + \operatorname{rank} X_2 - 1$.

Thus, by Theorem 1, (X', S') satisfies all conditions of optimality. □

Proof of Corollary 2. We now have a solution X' to $L(G')$ of rank rank $X_1 +$ rank $X_2 - 1$. Consider X' in the form $X' = \begin{bmatrix} Y_1 & y_1 & y_1 y_2^T \\ y_1^T & 1 & y_2^T \\ y_2 y_1^T & y_2 & Y_2 \end{bmatrix}$ again. The optimization problem to solve for X' involves maximizing the dot of X' and $L(G')$. Thus, modifying any entry of the upper right or lower left blocks in the matrix will not change the optimal value. It is known that we are able to complete these remaining values of the block matrix to a semidefinite matrix of rank equal to $\max(\operatorname{rank} X_1, \operatorname{rank} X_2)$, which gives the same objective value as X' and is thus still optimal. □

This corollary means that any graph which can be expressed as a vertex sum of two cliques can be analyzed in terms of its two summed components, and this analysis is no harder than solving the max-cut problem simply for the two components. Thus, we can reduce our analysis to biconnected graphs.

5.4. **Edge sum of two triangles.**

Proof of Theorem 4. The rank 1 solutions to the max-cut SDP on the smaller subgraphs give optimal solutions for the max-cut problem on such graphs. By Corollary 1, the optimal solutions are unique. Thus, the optimal solution to the max-cut problem on the graph G is unique and agrees with the solutions on the subgraphs G_1 and G_2.

Suppose now that $w_{23} \leq \min(\frac{1}{1/w_{12}+1/w_{13}}, \frac{1}{1/w_{24}+1/w_{34}})$. Let G_1' be the graph on 4 vertices with $w_{12}' = w_{12}, w_{13}' = w_{13}$, and $w_{23}' = \frac{1}{2} w_{23}$, and all other weights 0. Similarly, let G_2' with $w_{24}' = w_{24}, w_{34}' = w_{34}$, and $w_{23}' = \frac{1}{2} w_{23}$, and all other weights 0. Then, by Theorem 2, the vector $x = [-1, 1, 1, -1]$ is an optimal solution to the max-cut problem on both G_1' and G_2'. By Corollary 1, this solution is unique on both G_1' and G_2'. Thus, the matrix xx^T is indeed the unique optimal solution for the max-cut SDP on the full graph G.

On the other hand, if we know that xx^T is a solution to the max-cut SDP for G where $x = [-1, 1, 1 - 1]$, then x is in fact a solution to the max-cut problem on each of G, G_1, and G_2. Since we are given that G_1 and G_2 are both rank 1, then by Theorem 2, one of the weights $2_{12}, w_{13}$ and w_{23} has a reciprocal greater than the sum of the reciprocals of the other two

terms. Corollary 1 tells us that this is the unique weight such that $w_{ij}w_iw_j$ is positive. Since all w_{ij} are positive and $x_2x_3 = 1$, this weight is w_{23}. As a result,

$$\frac{1}{w_{23}} < \frac{1}{w_{12}} + \frac{1}{w_{13}}.$$

Now, suppose that xx^T is a primal solution, where $x = [-1, 1, 1, -1]$. Then The dual solution S^* is the unique dual solution which agrees with $L(G, w)$ on the off-diagonal values, and satisfies $S^*x = 0$. Note that

$$S^* = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S_2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -w_1 & w_1 & 0 \\ 0 & w_1 & -w_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\square$

satisfies both of these conditions, and thus must be the unique dual solution.

Proof of Theorem 5. Suppose for the sake of simplicity that $w_{12} > w_{13}$ and $w_{24} > w_{34}$. Then first assume the matrix $X = xx^T$, where $x = [\epsilon_1, -1, 1, \epsilon_2]$, is a solution. Repeating the argument from the previous proof, the vector x must be a solution to the max-cut problem for each of G, G_1 and G_2. By exhaustively checking all cases to solve the max-cut problem on G_1 and G_2, we find that $x = [1, -1, 1, 1]$. Now, the dual matrix has the form $S' = L(G, \bar{w})$, where $\bar{w}_{ij} = -x_ix_jw_{ij}$ by Lemma 1. The upper 3 by 3 principal sub matrix of S' has entries

$$\begin{bmatrix} w_{12} - w_{13} & -w_{12} & w_{13} \\ -w_{12} & w_{12} + w_{23} + w_{14} & -w_{23} \\ w_{13} & -w_{23} & -w_{13} + w_{23} - w_{34} \end{bmatrix}$$

The determinant of this matrix simplifies to

$$w_{23}(w_{12}-w_{13})(w_{24}-w_{34})-w_{12}w_{13}w_{24}+w_{12}w_{13}w_{34}-w_{12}w_{23}w_{24}+w_{13}w_{14}w_{24}.$$

Since S' is positive semidefinite, then this determinant must be nonnegative. For the determinant to be positive, w_{23} must satisfy

$$w_{23} \geq \frac{1}{1/w_{12} - 1/w_{13}} + \frac{1}{1/w_{24} - 1/w_{34}},$$

as desired.

On the other hand, if the weights satisfy both inequalities given in the first condition, then we can see for some $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ it is true that when $x = [\epsilon_1, -1, 1, \epsilon_2]$, then $X = xx^T$ is an optimal solution to any graph on 4 vertices such that it has 3 edges which form a triangle and the weights satisfy the condition in Theorem 2. Since we can find positive real numbers p and q such that $p + q = w_{23}$, $p \geq \frac{1}{1/w_{12}-1/w_{13}}$, and $q \geq \frac{1}{1/w_{24}-1/w_{34}}$. Thus, since X is an optimal solution to both of these graphs, it is an optimal solution for the large graph as well.

Note that again, similar to above, the existence of the primal optimal solution allows us to exactly find the dual solution of the matrix. We can check that

$$S^* = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S_2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & w_1 & w_1 & 0 \\ 0 & w_1 & w_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

indeed agrees with the Laplacian matrix on the off-diagonal values and that $S^*x = 0$, thus it must be the unique dual solution.   □

6. Future work

In our project, we began by considering a vertex sum, or what happened when we joined two subgraphs at a vertex. We were able to fully characterize a primal and dual solution to the SDP given solutions to the SDPs for the subgraphs. Doing the same for an edge sum, or joining two graphs at an edge, proved to be much more difficult. We were able to show a result for joining two K_3s given a constraint on the rank. We would like to further analyze the max-cut SDP for general edge sums, as well as for clique sums of larger cliques. To account for more general clique sums, one possible direction to look at is to add two graphs. In particular, the weight of an edge in this sum would be the sum of the corresponding edge weights of the summand graphs. This area is promising because it gave a good characterization for the theorem on the butterfly graph and for vertex sums.

A conjecture we have proposed for the solutions to the max-cut SDP of the edge sum of two graphs is as follows.

Conjecture 1. Let G_1 and G_2 be two graphs with vertex sets $V_1 = \{1, 2, \dots, m+1, m+2\}$, and $V_2 = \{m+1, m+2, \dots, n-1, n\}$. Let (X_1, S_1) and (X_2, S_2) be the primal-dual solution pairs to the max-cut SDPs on G_1 and G_2, respectively.

Let G be the edge sum of G_1 and G_2, with common edge $(m+1, m+2)$. Let (X^*, S^*) be the primal and dual optimal solutions to the graph.

Then, the following two statements are equivalent:

(1) X_1, X_2 are rank 1 matrices and agree on the intersection.

(2) $S = \begin{bmatrix} S_1^* & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S_2^* \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & w_1 & \pm w_1 & 0 \\ 0 & \pm w_1 & w_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. for some common

choice of $\pm$ on the off-diagonal w_1s. The first two matrices are $n \times n$. The matrix S_1 is $m+2 \times m+2$, and the matrix S_2 is $n-m \times n-m$. In the last matrix, the w_1s are in entry $A_{m+1,m+1}$ and $A_{m+2,m+2}$.

In addition, we would like to extend our analysis of SDP solutions in the future to cover more families of graphs and operations upon them. In particular, one family of graphs we would like to potentially investigate in

the future are series-parallel graphs. Series-parallel graphs can easily be decomposed into vertex sums, which means we may be able to generally characterize their rank one solutions.

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