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Title of Research Report

A Turán-Type Problem in Mixed Graphs

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A TURÁN-TYPE PROBLEM IN MIXED GRAPHS

EDWARD YU

ABSTRACT. A *mixed graph* is a graph-like object that comprises a collection of vertices and edges which connect pairs of vertices; edges can be either directed or undirected. We study a natural Turán-type problem on mixed graphs, seeking to understand how large a fraction of directed edges an F-free mixed graph can have. We consider a *Turán density coefficient* on mixed graphs, and show that it has many properties analogous to classical Turán numbers in graphs and hypergraphs.

Such properties enable us to establish an analogue of the Erdős-Stone-Simonovits theorem and give a variational characterization of the Turán density coefficient of any mixed graph (along with an associated extremal F-free family). This characterization enables us to highlight an important divergence between classical extremal numbers and the Turán density coefficient; we exhibit a mixed graph for which the Turán density coefficient is irrational. However, unlike the case of Turán-type problems on multigraphs, we show that Turán density coefficients are always algebraic.

KEYWORDS. extremal combinatorics, graphs, mixed graphs, Turán density, Zykov symmetrization

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The participating team declares that the paper submitted is comprised of original research and results obtained under the guidance of the instructor. To the team's best knowledge, the paper does not contain research results, published or not, from a person who is not a team member, except for the content listed in the references and the acknowledgment. If there is any misinformation, we are willing to take all the related responsibilities.

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1. Introduction

1.1. **Motivation.** Extremal combinatorics is a rich subfield of combinatorics which studies how large (or small) an object satisfying certain constraints may be. Some of the primary objects of study are graphs, networks that are comprised of *nodes* or *vertices*, some pairs of which are connected by *edges*. Problems in extremal graph theory are exciting on their own merits, but often also have wide ranging interest to the mathematical community and applications that reach far beyond mathematics into computer science, electrical engineering, voting theory, physics, and much more; some especially famous results of note include Turán's theorem [35] and the computer-assisted proof of the four-color theorem [28]. However, many very fundamental (and simple-sounding) questions remain wide open despite tremendous efforts to make progress (e.g. [34, 16, 30, 17]).

One of the earliest results in extremal graph theory was Mantel's theorem in 1907 [23]; Mantel showed that an n-vertex graph which does not contain a triangle has at most $\frac{n^2}{4}$ edges. More generally, a fundamental family of questions in the field of extremal graph theory are *Turán-type questions*, which seek to understand how large a graph-like object can be when we forbid certain substructures from appearing. We make the above notions more precise below.

Definition 1.1. A graph $G = (V(G), E(G))$ is comprised of a *vertex set* $V(G)$ of size $v(G)$ and *edge set* $E(G)$ consisting of $e(G)$ unordered pairs of vertices.

Given a fixed graph F, we say that F is a *subgraph* of graph G (denoted $F \subseteq G$) if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$; in other words, we can obtain F from G by deleting vertices and edges. If F is not a subgraph of G, we say that G is *F-free*.

Turán-type problems are named in part due to a famous theorem of Turán from 1941 [35] which generalized Mantel's theorem to *complete graphs*, graphs where all pairs of vertices are connected by edges (the n-vertex complete graph is denoted K_n); Erdős, Stone, and Simonovits gave an asymptotic result for all graphs in [12, 13], sometimes described as "the fundamental theorem of extremal graph theory":

Theorem 1.2 (Erdős-Stone-Simonovits, 1946)**.** *For graph F and positive integer n, let the* extremal number $\mathrm{ex}(n, F)$ *be the maximum number of edges in a n-vertex F-free graph. Then, as $n \to \infty$ we have that*

$$\mathrm{ex}(n, F)/\binom{n}{2} = \frac{\chi(F) - 2}{\chi(F) - 1} + o(1),$$

where $\chi(F)$ denotes the chromatic number *of F, the minimum number of colors required in a proper vertex-coloring of F (where no two adjacent vertices are the same color).*

Other Turán-type problems include the rainbow Turán problem (on edge-colored graphs), the directed-graph Turán problem, and the hypergraph Turán problem; the last of these in particular is a rich and fertile area of research which has attracted much attention. Major known Turán results are further discussed in Section 2.1.

1.2. **Mixed graphs.** A *mixed graph* is a graph with both undirected and directed edges (see Figure 2 for illustrative examples). In this work, we are interested in studying Turán-type questions on mixed graphs, motivated by applications to theoretical computer science and other graph theoretic interest (see Section 2.2 for details). To precisely state the problem we study, we first define appropriate notions of edge counts and subgraphs for mixed graphs:

Definition 1.3. A *mixed graph* is comprised of a vertex set V and an edge set E. Each edge $e \in E$ links two different vertices $u, v \in V$, and can either be *undirected* ($e = uv = vu$) or *directed* ($e = u\check{v} = \check{v}u$). For directed edge $u\check{v}$, v is the *head* vertex of the edge, and u the *tail*. No two edges can connect the same pair of vertices.

For mixed graph G, let $e_{\mathrm{u}}(G)$ and $e_{\mathrm{d}}(G)$ be the number of undirected and directed edges respectively, and let $\alpha(G) = e_{\mathrm{u}}(G)/\binom{n}{2}$ and $\beta(G) = e_{\mathrm{u}}(G)/\binom{n}{2}$ be the undirected and directed edge densities, respectively. For vertex $v \in V(G)$ define its *undirected degree* $\deg_{\mathrm{u}} v$ as the number of vertices connected to v by undirected edges, and its *directed degree* $\deg_{\mathrm{d}} v$ as the number of vertices connected to v by directed edges.

Definition 1.4. We say that the mixed graph F is a *subgraph* of G ($F \subseteq G$) if one can obtain F from G by deleting vertices, deleting edges, and forgetting edge directions. We say that G is *F-free* if F is not a subgraph of G.

Example 1.5. In Figure 1, both F_1 and F_2 are subgraphs of G; F_1 is not a subgraph of F_2, or vice versa.

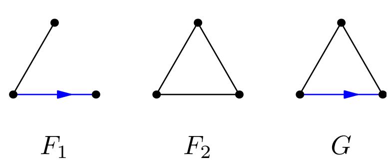


Figure 1

The above definition of subgraphs is motivated by applications such as [11]. One might initially consider the "naive" extremal problem, to maximize the edge density $\alpha(G) + \beta(G)$ over all F-free G. This is a valid formulation for undirected graphs, because there exists a unique notion of maximal edge count, and thus edge density. However, this is uninteresting for mixed graphs—for all F with at least one directed edge, $F \not\subseteq K_n$ and $\alpha(K_n) + \beta(K_n) = 1$, hence the maximal value of $\alpha(G) + \beta(G)$ over F-free G would simply be 1.

This motivates a definition of a more interesting quantity. We may imagine weighting the directed edges, so that each contributes some constant $\rho \geq 1$ to the edge count (while undirected edges still contribute 1), so our goal is to maximize $\alpha(G) + \rho\beta(G)$. Clearly when $\rho = 1$ we have $\alpha(G) + \rho\beta(G) \leq 1$, and as ρ increases, the maximal value of $\alpha(G) + \rho\beta(G)$ is nondecreasing. Hence, there intuitively should exist a value of ρ for which the maximal value of $\alpha(G) + \rho\beta(G)$ is at most 1 below this value, and greater than 1 above it. We call this value the *Turán density coefficient*, $\theta(F)$. (We defer a formal proof of the existence of $\theta(F)$ to Proposition 2.2.)

Definition 1.6. Let F be a mixed graph. We define the *Turán density coefficient* $\theta(F)$ as the maximum ρ such that

$$\alpha(G) + \rho\beta(G) \leq 1 + o_{n\to\infty}(1)$$

over all F-free n-vertex mixed graphs G. If $\beta(G) = o(1)$ over F-free n-vertex G, we say $\theta(F) = \infty$.

The above quantity naturally arises in various theoretical computer science problems; for example, Dong, Mani, and Zhao studied a generalization of the above quantity for a specific family of mixed hypergraphs F in the context of the k-SAT enumeration problem [11].

1.3. **Main results.** Our first theorem gives a result for mixed graphs with at most one directed edge which is analogous to the Erdős-Stone-Simonovits Theorem.

Let $\widetilde{F}$ be the *underlying undirected graph* of F, obtained by forgetting the directions of all directed edges (but retaining the edges).

Theorem 1.7. *If F is a mixed graph with at most one directed edge, then $\theta(F) = 1 + \frac{1}{\chi\left(\widetilde{F}\right)-2}$ (if $\chi\left(\widetilde{F}\right) \leq 2$ then $\theta(F) = \infty$).*

See Example 3.3 for an example. The general case is more nuanced, since mixed graphs have much more complicated extremal behavior than undirected graphs (a notion we make precise in several ways in the subsequent results). This independently motivates their study, given the vast literature and interest in identifying surprising, complicated, or otherwise unusual phenomena that arise in studying extremal problems in graph-like objects.

As a starting point, we can classify mixed graphs as either *collapsible* or *uncollapsible* (see Definition 3.4), and arrive at the following bounds on $\theta(F)$ for a general mixed graph F.

Theorem 1.8. *Let F be a mixed graph with at least one edge (directed or undirected). Then*

- *$\theta(F) = 1$ if and only if F is uncollapsible;*
- *$\theta(F) = \infty$ if and only if F is collapsible and $\chi\left(\widetilde{F}\right) = 2$;*

- *otherwise,*

$$1 + \frac{1}{\chi\left(\widetilde{F}\right)} \leq \theta(F) \leq 1 + \frac{1}{\chi\left(\widetilde{F}\right) - 2}.$$

It is possible to exhibit mixed graphs which achieve both ends of the inequality: the upper bound is attained by Theorem 1.7, and an example where the lower bound is tight is given in Example 3.12. Hence, the inequality in Theorem 1.8 is tight on both ends, though unfortunately we cannot determine the exact value of $\theta(F)$ from this inequality in general. Towards more precisely understanding $\theta(F)$, we first study in Section 4 mixed adjacency matrices (defined in Definition 4.2) and obtain a result bearing strong resemblance to Theorem 1 in [3], by Brown, Erdős, and Simonovits on Turán-type problems in directed graphs. We then use this result to give a variational characterization in Theorem 5.3 which determines $\theta(F)$ as the solution to a finite-dimensional optimization problem.

These results suggest that $\theta(F)$ is a much more complicated object than the classical extremal numbers of graphs. While the Erdős-Stone-Simonovits theorem implies that graphs have extremal numbers that are always of the form $1 - \frac{1}{k}$ (for positive integer k), mixed graphs exhibit a markedly different behavior, as the following result highlights by example (constructing a mixed graph F with $\theta(F) = 1 + \frac{1}{\sqrt{2}}$):

Theorem 1.9. *There exists a mixed graph F for which $\theta(F)$ is irrational.*

Based on the above, one might wonder how complicated $\theta(F)$ can be. The complexity of Turán-type quantities in discrete structures varies considerably. Turán numbers of graphs are always rational, as noted above; Turán numbers of families of hypergraphs can have arbitrarily high algebraic degree [21, 27]; and families of multigraphs can even have transcedental extremal numbers [24]. We leverage our variational characterization to establish algebraicity of mixed graph Turán density coefficients, a notion we make more precise below.

Theorem 1.10. *Let F be a mixed graph such that $\theta(F) < \infty$. Then $\theta(F)$ is an algebraic number.*

The following conjecture is similar to Theorem 1.2 in [21] and invites further investigation:

Conjecture 1.11. *There exist finite families of mixed graphs with Turán density coefficients of arbitrarily high algebraic degree. (The Turán density coefficient of a family is defined in the same manner as that for a single mixed graph, except all mixed graphs in the family are forbidden.)*

Organization. In Section 2 we give additional background on Turán results and summarize some previous work on mixed graphs. We proceed in Section 2.3 to establish fundamental properties of the Turán density coefficient we define, including supersaturation [14] and blowup lemmas [17]. We then continue in Section 2.4 to determine $\theta(F)$ for some simple familes of F. These preliminary results prepare us to prove an analogue of the Erdős-Stone-Simonovits theorem in Section 3. We first investigate the case where F has at most one directed edge; here an exact asymptotic result can be given in terms of $\chi\left(\widetilde{F}\right)$. We then apply this result to give an inequality which holds for all mixed graphs. We give examples of mixed graphs on both ends of the inequality, showing it is tight.

The primary goal of the later sections is to develop a variational characterization for $\theta(F)$ and apply it to obtain additional results determining the precise value of $\theta(F)$. We begin this analysis in Section 4 by investigating a seemingly different extremal problem on mixed graphs: for fixed ρ, maximize $\alpha(G) + \rho\beta(G)$ over F-free G. Following work in [3] we show that there exist asymptotically extremal mixed graphs that arise as asymmetric blowups of relatively small mixed graphs. The work done on this extremal problem is translated into a variational characterization in Section 5. In Section 6, we concern ourselves with the possible values that $\theta(F)$ can take: we first exhibit a mixed graph F where $\theta(F)$ is irrational in Section 6.1, and then show that $\theta(F)$ is always an algebraic number in Section 6.2 (giving a bound on the algebraic degree).

2. Preliminaries

Notation. We use $\mathbb{N}$ to denote the set of nonnegative integers. For positive integer r, $[r]$ is the set $\{1, \dots, r\}$.

We will use $\mathbb{E}[X]$ to denote the expected value of random variable X. We will use $\mathbb{P}[A]$ to denote the probability of event A.

For vector $\mathbf{x} = (x_1, \dots, x_r)$ let $\|\mathbf{x}\|_1 := \sum_{i=1}^{r} |x_i|$ denote its ℓ^1 norm. We use $\mathbf{1}$ to denote the vector with all coefficients 1 (size will be clear from context).

We use $\triangle^{r-1}$ to denote the $(r-1)$-dimensional simplex

$$\triangle^{r-1} = \{\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{R}^r \mid \|\mathbf{x}\|_1 = 1, x_i \geq 0 \text{ for all } i\}.$$

For $m \times n$ matrix A, we use A_{ij} to denote the entry in the ith row and jth column of A. For sets S, T we use $S \sqcup T$ to denote the disjoint union of S and T. Asymptotic notation, such as $o(1)$, will always refer to the limit for large n.

2.1. **Overview of Turán results.** Graph or hypergraph Turán problems (and variants thereof) are one of the most studied problems in combinatorics, and are the source of several major open questions.

One is the bipartite Turán problem: while the Erdős-Stone-Simonovits theorem appears to give a complete asymptotic characterization for Turán problem on graphs, it in fact gives only $\mathrm{ex}(n, F) = o(n^2)$ for bipartite graphs. The problem of finding the true exponent for bipartite graphs is still unsolved. The Kővári-Sós-Turán theorem [19] bounds the extremal number of $K_{s,t}$, the complete bipartite graph with part sizes s, t, by $\mathrm{ex}(n, K_{s,t}) \leq O(n^{1-1/s})$

for $s \leq t$; this is is known to be tight in some cases [6], and is conjectured to be tight always, but this is still wide open.

Recent attention has also been focused on the hypergraph Turán problem, where even the density of the tetrahedron K_4^3 is unknown; the known results in this field are a scattered assortment of graphs (of which the Fano plane is one) [17, 7]. Many powerful techniques based on probabilistic methods, homomorphism counting, and flag algebras have been developed through the study of these Turán problems, which are currently one of the most active subfields of graph theory.

Several variants of Turán problems on graphs have also been studied. One is the directed graph problem, studied by Erdős, Simonovits, Brown, and Harary, among others (see [4, 3, 38]). Others include the multigraph problem [5], where multiple edges are allowed between pairs of vertices, and the rainbow Turán problem [18], where edges of the forbidden graph are colored to impose an additional restraint.

Extremal constructions of Turán-type problems are also of general graph-theoretic interest. For instance, the *Turán graph* $T(n, r)$ is the unique K_{r+1}-free graph on n vertices with the maximal number of edges: it is formed by partitioning the n vertices into r subsets of sizes as equal as possible, then adding edges between vertices in different subsets. This special structure makes the Turán graph an interesting object in its own right; similarly, the mixed graphs that arise as extremal in our analysis are natural objects of study.

2.2. **Previous work on mixed graphs.** Mixed graphs as graph-theoretic objects also have been studied in a variety of contexts—spectral graph theory on mixed graphs was developed in [20, 2, 29] using concepts of mixed adjacency matrices similar to that in Definition 4.2. In [22] bounds are given for the extremal degree-diameter problem of finding the maximal number of edges in a mixed graph with limited maximum degree and diameter.

Mixed graphs also arise naturally in the context of theoretical computer science, including the extremal problems of coloring [31, 26, 15] and job scheduling [32]. Mixed graphs are also applied in the context of neural networks, since they are highly useful in encoding knowledge relationships in large-scale networks, in particular ones where nodes are linked by possibly-bidirectional notions of correlation or causality. They have numerous applications in object classification and labeling [36, 10, 25], social network models [33], and inference on Bayesian networks. Similar extremal problems also exist in logic programming: mixed graphs and more general objects are relational structures which have been used in a variety of problems in complexity theory and propositional logic [8, 9].

2.3. **Mixed graph fundamentals.** Recall that a mixed graph is a simple graph where edges can either be undirected or directed.

Example 2.1. In Figure 2, the first two graphs are examples of mixed graphs (note that a fully undirected graph is still a mixed graph); the last two are not (self-loops and vertices connected by multiple edges are disallowed).

Figure 2

Observe that $\theta(F) \geq 1$ for all mixed graphs F, and that if $F \subseteq G$ then $\theta(F) \geq \theta(G)$.

We begin by confirming that $\theta(F)$ is actually well-defined via straightforward analysis.

Proposition 2.2 (Existence of $\theta(F)$)**.** *For any mixed graph F, either $\beta(G) = o(1)$ over all F-free n-vertex mixed graphs G, or there exists a maximal value of ρ such that*

$$\limsup_{\substack{F \not\subseteq G \\ v(G) \to \infty}} \alpha(G) + \rho\beta(G) \leq 1.$$

Proof. Let

$$f(\rho) := \limsup_{\substack{F \not\subseteq G \\ v(G) \to \infty}} \alpha(G) + \rho\beta(G),$$

so it is clear that $f(\rho)$ is nondecreasing, and $0 \leq f(\rho) \leq \max(1, \rho)$ for all $\rho \in (0, \infty)$. Thus, it suffices to show that $f(\rho)$ is continuous. Since $\beta(G) \leq 1$ for any mixed graph G, for all $\epsilon > 0$ we have that

$$f(\rho) \leq f(\rho + \epsilon) = \limsup_{\substack{F \not\subseteq G \\ v(G) \to \infty}} \alpha(G) + (\rho + \epsilon)\beta(G) \leq \limsup_{\substack{F \not\subseteq G \\ v(G) \to \infty}} \alpha(G) + \rho\beta(G) + \epsilon = f(\rho) + \epsilon,$$

and similarly $f(\rho - \epsilon) \in [f(\rho) - \epsilon, f(\rho)]$. Continuity follows immediately. $\square$

Next, we show analogues of classical supersaturation and blowup density results similar to those in [14, 17], using probabilistic methods.

Definition 2.3. Let G be a mixed graph and $S \subseteq V(G)$. The *induced subgraph* $G[S]$ is the mixed graph with vertex set S and edge set consisting of all edges in G with both endpoints in S.

Lemma 2.4 (Supersaturation)**.** *Given a mixed graph F for which $\theta(F) < \infty$ and $\epsilon > 0$, there exists some constant $c = c(\epsilon) > 0$ such that any n-vertex mixed graph G satisfying $\alpha(G) + \theta(F)\beta(G) \geq 1 + \epsilon$ must contain at least $c \cdot n^{v(F)}$ copies of F for n sufficiently large.*

Proof. Choose N_0 such that any mixed graph H on N_0 vertices satisfying $\alpha(H)+\theta(F)\beta(H) \geq 1 + \frac{\epsilon}{2}$ must contain F as a subgraph. Let $n > N_0$ and G be any mixed graph on n vertices with $\alpha(G)+\theta(F)\beta(G) \geq 1+\epsilon$. Let S be a subset of N_0 vertices from $V(G)$ selected uniformly at random, so

$$\mathbb{E}\big[\alpha(G[S]) + \theta(F)\beta(G[S])\big] = \alpha(G) + \theta(F)\beta(G) \geq 1 + \epsilon. \tag{2.1}$$

Claim 2.5. If X is a random variable in $[0,l]$ then $\mathbb{P}\big[X \geq \mathbb{E}[X] - \epsilon\big] \geq \frac{\epsilon}{l}$ for all $\epsilon > 0$.

Proof. Otherwise, let $\delta = \mathbb{P}\big[\mathbb{E}[X]-\epsilon \leq X \leq l\big]$, so $\delta < \frac{\epsilon}{l}$. Then $\mathbb{P}\big[0 \leq X < \mathbb{E}[X]-\epsilon\big] = 1-\delta$, and

$$\mathbb{E}[X] \leq l \cdot \delta + (\mathbb{E}[X] - \epsilon)(1-\delta) < \epsilon + (\mathbb{E}[X] - \epsilon) = \mathbb{E}[X],$$

contradiction. □

Applying the claim to (2.1), we find that

$$\mathbb{P}\left[\alpha(G[S]) + \theta(F)\beta(G[S]) \geq 1 + \frac{\epsilon}{2}\right] \geq \frac{\epsilon}{2\theta(F)}. \tag{2.2}$$

When S satisfies $\alpha(G[S]) + \theta(F)\beta(G[S]) \geq 1 + \frac{\epsilon}{2}$, $G[S]$ must contain a copy of F as a subgraph. Thus, if $T \subseteq S$ is a random subset of size $v(F)$, then

$$\mathbb{P}\big[F \subseteq G[T]\big] \geq \frac{1}{\binom{N_0}{v(F)}}. \tag{2.3}$$

Combining (2.2) and (2.3), we know that if T is a random subset of $V(G)$ of size $v(F)$, then

$$\mathbb{P}\big[F \subseteq G[T]\big] \geq \frac{\epsilon}{2\theta(F)} \cdot \frac{1}{\binom{N_0}{v(F)}}.$$

Thus the number of copies of F in G is at least

$$\frac{\epsilon}{2\theta(F)} \cdot \frac{1}{\binom{N_0}{v(F)}} \cdot \binom{n}{v(F)},$$

and the conclusion follows. □

Definition 2.6. Given a mixed graph F and an integer $t \geq 2$, let $F[t]$ denote the *balanced t-blowup* of F, obtained by replacing each vertex $v_i \in V(F)$ by t copies $v_{i,1}, \ldots, v_{i,t}$, and each undirected edge v_iv_j with t^2 undirected edges $v_{i,k}v_{j,l}$ $(1 \leq k, l \leq t)$, and each directed edge $v_i\check{v}_j$ with t^2 directed edges $v_{i,k}\check{v}_{j,l}$ $(1 \leq k, l \leq t)$.

Example 2.7. Figure 3 is an illustration of a balanced 2-blowup of a mixed graph.

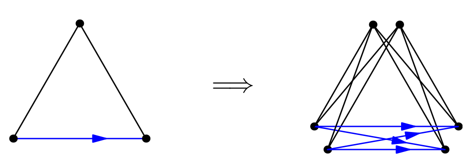


Figure 3

Lemma 2.8 (Blowups)**.** *Given integer $t \geq 1$ and a mixed graph F with $\theta(F) < \infty$, $\theta(F) = \theta(F[t])$.*

Proof. Suppose otherwise; then there is an integer $t \geq 2$ and a mixed graph F on r vertices such that $\theta(F) < \infty$ and $\theta(F) \neq \theta(F[t])$. Since $F \subseteq F[t]$, $\theta(F) > \theta(F[t])$. Then, there is $\epsilon > 0$ such that there exist $F[t]$-free graphs G on n vertices for arbitrarily large n which satisfy

$$\alpha(G) + \theta(F)\beta(G) > 1 + \epsilon,$$

otherwise we would have $\alpha(G) + \theta(F)\beta(G) \leq 1 + o(1)$, which would mean $\theta(F) \leq \theta(F[t])$. Then by Lemma 2.4, G contains at least cn^r copies of F. Randomly select an equitable partition of the vertex set of G into parts $V_1, \dots, V_r$, i.e. $\left\lfloor \frac{n}{r} \right\rfloor \leq |V_i| \leq \left\lceil \frac{n}{r} \right\rceil$ for all i. Hence for any copy of F in G, the probability that its i-th vertex is in V_i is $\frac{|V_i|}{n} \geq \frac{\left\lfloor \frac{n}{r} \right\rfloor}{n} > \frac{1}{2r}$. Thus the probability that any given copy of F has its i-th vertex in V_i (for all i) is greater than $1/(2r)^r$. This implies there exists a partition $\{V_1, \dots, V_r\}$ of the vertex set of G such that G contains at least $c\left(\frac{n}{2r}\right)^r = \frac{c}{(2r)^r}n^r$ copies of F where each copy has its i-th vertex in V_i for all i.

Now we construct a r-partite, r-uniform hypergraph H on the vertex set of G, such that for $v_i \in V_i, (i \in [r])$, $(v_1, \dots, v_r)$ is an edge in H if and only if $\{v_1, \dots, v_r\}$ forms a copy of F in G. Then H has at least $\frac{c}{(2r)^r} \cdot n^r$ edges by construction. By Theorem 2.2 of [17]: because the Turán density of the single r-uniform hyperedge K_r^r is clearly zero, the density of $K_r^r[t]$ is zero as well. Hence, H contains a copy of $K_r^r[t]$ as a subgraph when n is sufficiently large, and thus $F[t] \subseteq G$, a contradiction. □

2.4. **Simple families of mixed graphs.** We establish our first results on $\theta(F)$, for some simple families F. These will also be useful later on in Section 3.

We first deal with bipartite mixed graphs, i.e. mixed graphs F for which $\chi\left(\widetilde{F}\right) = 2$. For positive integer a, b, let $K_{a,b}$ denote the undirected complete bipartite graph with a vertices in one part and b vertices in the other. Let $K_{\overrightarrow{a,b}}$ denote the directed complete bipartite mixed graph with a vertices in one part and b vertices in the other, where all edges are directed from the first part towards the second part. Just as the Erdős-Stone-Simonovits theorem guarantees $\mathrm{ex}(n, K_{a,b}) = o(n^2)$, we claim that $\theta(K_{\overrightarrow{a,b}}) = \infty$,

Proposition 2.9. *Let F be a mixed graph such that $F \subseteq K_{\overrightarrow{t,t}}$ for some t. Then $\theta(F) = \infty$.*

Proof. It suffices to prove this for $F = K_{\overrightarrow{t,t}}$ itself. We show that for any $c > 0$, for mixed graph G on n vertices, if $\beta(G) > c$ then $K_{\overrightarrow{t,t}} \subseteq G$ as long as n is sufficiently large.

Choose $n' \gg t$ and $n \gg n'$. First, delete all undirected edges from G to form G_1 which has more than $c\binom{n}{2}$ edges, all directed. By the Erdős-Stone-Simonovits Theorem for $K_{n',n'}$ on the supergraph $\widetilde{G_1}$, there exists some subgraph $G_2 \subseteq G_1$ such that $\widetilde{G_2} = K_{n',n'}$ since n is sufficiently large. Since G_2 is bipartite, partition $V(G_2) = A \sqcup B$; without loss of generality assume that more edges in G_2 have their head in A than B, and delete all edges with heads in B. The resulting graph G_3 is a fully directed bipartite graph with all edges pointing from B to A, with edge density at least $\frac{1}{4} > 0$. Since n' is sufficiently large, we may apply the Erdős-Stone-Simonovits Theorem for $K_{t,t}$ on the supergraph $\widetilde{G_3}$ which gives $K_{\overrightarrow{t,t}}$ as a subgraph of G. ◻

We can also easily find $\theta(F)$ when F has no directed edges by applying the Erdős-Stone-Simonovits theorem.

Proposition 2.10. *Let F be an undirected graph with chromatic number $\chi(F)$. Then*

$$\theta(F) = \begin{cases} 1 + \frac{1}{\chi(F)-2}, & \text{if } \chi(F) > 2, \\ \infty, & \text{if } \chi(F) \leq 2. \end{cases}$$

Proof. The $\chi(F) \leq 2$ case is resolved by Proposition 2.9; assume $\chi(F) \geq 3$ in the remainder of the proof. Note that G is F-free if and only if its underlying undirected graph $\widetilde{G}$ is F-free. Hence for all F-free G we have $e_{\mathrm{u}}(G) + e_{\mathrm{d}}(G) = e\big(\widetilde{G}\big) \leq \mathrm{ex}(n, F) \leq \left(\frac{\chi(F)-2}{\chi(F)-1} + o(1)\right)\frac{n^2}{2}$ by the Erdős-Stone-Simonovits theorem. Now, compute

$$\alpha(G) + \frac{\chi(F)-1}{\chi(F)-2}\beta(G) \leq \frac{\chi(F)-1}{\chi(F)-2}\left(\alpha(G)+\beta(G)\right) = \frac{\chi(F)-1}{\chi(F)-2}\left(\frac{e_{\mathrm{u}}(G) + e_{\mathrm{d}}(G)}{\binom{n}{2}}\right) \leq 1 + o(1).$$

Equality holds when G is the Turán graph $T(n, \chi(F)-1)$ with all edges directed arbitrarily (which is indeed F-free), so $\theta(F) = \frac{\chi(F)-1}{\chi(F)-2}$. ◻

3. Proof of Theorem 1.7 and Theorem 1.8

3.1. **Mixed graphs with one directed edge.** This subsection first gives a precise result for complete mixed graphs with one directed edge which holds for all n-vertex F-free graphs, analogous to Turán's theorem, in Proposition 3.1. We use the Zykov symmetrization technique [1] to constrain the structure of an F-free graph, then use this structural information to bound the edge count of the graph. Theorem 1.7 follows as a corollary.

We will use $t(n,r)$ for the number of edges in $T(n,r)$, the r-partite *Turán graph* on n vertices.

Proposition 3.1. *Let $n \geq r \geq 2$ be integers. For all n-vertex $\overrightarrow{K_{r+1}}$-free mixed graphs G, $\alpha(G) + \frac{\binom{n}{2}}{t(n,r)}\beta(G) \leq 1$.*

The proof is deferred to Appendix A.

Corollary 3.2. *For any $r \geq 3$, we have $\theta\left(\overrightarrow{K_r}\right) = 1 + \frac{1}{r-2}$.*

Proof. The lower bound is obtained by taking the limit in Proposition 3.1 as $n \to \infty$. The upper bound is obtained by construction: as before, let G be any graph with all edges directed and $\widetilde{G} = T(n, r-1)$; then G is F-free, $\alpha(G) = 0, \beta(G) = \frac{r-2}{r-1} + o(1)$, so

$$0 + \theta(\overrightarrow{K_r}) \cdot \left(\frac{r-2}{r-1} + o(1)\right) \leq 1.$$

This implies $\theta\left(\overrightarrow{K_r}\right) \leq 1 + \frac{1}{r-2}$, so $\theta\left(\overrightarrow{K_r}\right) = 1 + \frac{1}{r-2}$ as claimed. □

We now prove Theorem 1.7 using this result, and the blow-up density result, Lemma 2.8.

Proof of Theorem 1.7. The case where F has no directed edges is covered by Proposition 2.10, so we deal with the case where F has one directed edge. For brevity let $r = \chi\left(\widetilde{F}\right)$. If $r = 2$ then F is a subgraph of $K_{\overrightarrow{t,t}}$ for some t, so $\theta(F) = \infty$ by Proposition 2.9.

Henceforth assume $r > 2$, and take the vertex coloring of F to obtain chromatic parts $V(F) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_r$. Without loss of generality, we assume the directed edge of F is from a vertex in V_1 to a vertex in V_2. Let F' be the mixed graph obtained by making all edges between V_1 and V_2 directed with head vertex in V_2 (see Figure 4).

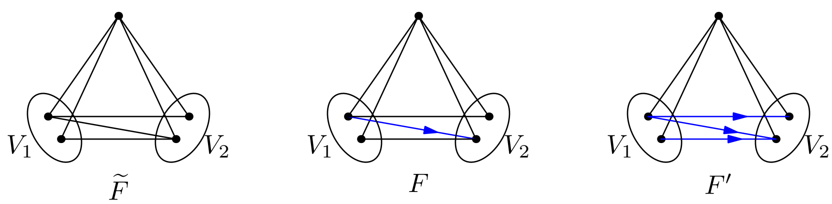


Figure 4

By Proposition 2.10 we have

$$\theta\left(\widetilde{F}\right) = 1 + \frac{1}{r-2}, \tag{3.1}$$

On the other hand, by Corollary 3.2, $\theta\left(\overrightarrow{K_r}\right) \geq 1 + \frac{1}{r-2}$. Notice that F' is a subgraph of a balanced t-blowup of $\overrightarrow{K_r}$ for sufficiently large t. Therefore by Lemma 2.8,

$$1 + \frac{1}{r-2} = \theta\left(\overrightarrow{K_r}\right) = \theta\left(\overrightarrow{K_r}[t]\right) \leq \theta(F'). \tag{3.2}$$

Finally, since $\widetilde{F} \subseteq F \subseteq F'$, we have $\theta(F') \leq \theta(F) \leq \theta\left(\widetilde{F}\right)$. Combining (3.1) and (3.2),

$$1 + \frac{1}{r-2} \leq \theta(F') \leq \theta(F) \leq \theta\left(\widetilde{F}\right) = 1 + \frac{1}{r-2}.$$

□

Example 3.3. In Figure 5 we have $\theta(F_1) = \infty$ and $\theta(F_2) = \frac{3}{2}$.

Figure 5

3.2. **General mixed graphs.** We will now concern ourselves with the general case. To obtain bounds on $\theta(F)$ for mixed graph F, we attempt to collapse F into a mixed graph with one directed edge, at which point we may apply Theorem 1.7. Earlier, we alluded to a notion of *collapsibility* for mixed graphs. This notion characterizes whether or not the aforementioned collapsing actually is possible—if not, the mixed graph is uncollapsible and can be shown to have a density coefficient of 1; if so, the mixed graph is collapsible and can be bounded.

We begin formalizing this notion of collapsibility.

Definition 3.4. We say a mixed graph is *uncollapsible* if some two head vertices are adjacent or some two tail vertices are adjacent. Otherwise, it is *collapsible*.

Example 3.5. Figure 6 shows three uncollapsible configurations.

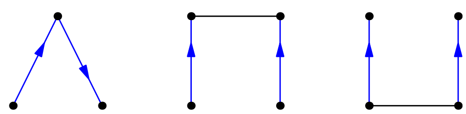


Figure 6

Let F be a collapsible mixed graph with at least one directed edge. Definition 3.4 implies that the vertex set of F can be partitioned into $V(F) = V_0 \sqcup V_h \sqcup V_t$ where V_0 are all vertices in F that are neither heads nor tails of any edge, V_h are head vertices of some edges, V_t are tail vertices of some edges; both V_h and V_t are independent sets.

Definition 3.6. The *head-tail collapsion* of F, denoted by $F^{\triangleright}$, is the mixed graph obtained by contracting V_h into a single vertex h and V_t into a single vertex t. More precisely, $V(F^{\triangleright}) = V_0 \cup \{h\} \cup \{t\}$, and $E(F^{\triangleright})$ consists of the directed edge $\check{h}t$, all undirected edges ab where $a, b \in V_0, ab \in E(F)$, all undirected edges ah where $a \in V_0$ and there exists $v \in V_h$ such that $av \in E(F)$, and all undirected edges at where $a \in V_0$ and there exists $v \in V_t$ such that $av \in E(F)$.

Example 3.7. Figure 7 illustrates an example of the head-tail collapsion.

Definition 3.8. For positive integer n and real number $0 < x < 1$, let $M(x, n)$ denote the following mixed graph with n vertices, constructed in three steps:

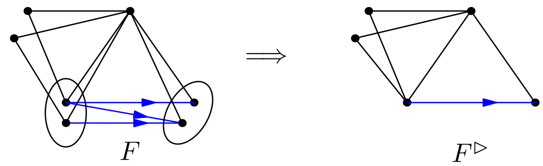


Figure 7

(i) Construct an undirected complete graph X on $\lfloor nx \rfloor$ vertices;
(ii) Construct an independent set Y with $\lceil n(1-x) \rceil$ vertices;
(iii) Draw a directed edge from every vertex in X towards every vertex in Y.

This construction is illustrated in Figure 8.

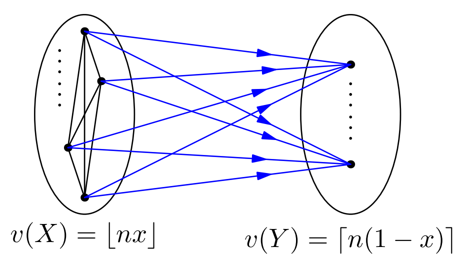


Figure 8

Proposition 3.9. *Let F be a mixed graph. If $F \not\subseteq M(x,n)$ for all $x \in (0,1)$ and positive integers n, then $\theta(F) = 1$.*

Proof. We will show that $\theta(F) \le 1+\epsilon$ for any small $\epsilon > 0$. Let $x = 1 - \frac{\epsilon}{1+2\epsilon} = \frac{1+\epsilon}{1+2\epsilon}$, and compute the edge densities of $M(x,n)$ as

$$\alpha\big(M(x,n)\big) = \frac{\binom{\lfloor nx \rfloor}{2}}{\binom{n}{2}} > \frac{\binom{nx-1}{2}}{\binom{n}{2}},$$
$$\beta\big(M(x,n)\big) = \frac{\lfloor nx \rfloor \cdot \lceil n(1-x) \rceil}{\binom{n}{2}} > \frac{(nx-1)n(1-x)}{\binom{n}{2}}.$$

Asymptotically,

$$\lim_{n\to\infty} \frac{\binom{nx-1}{2}}{\binom{n}{2}} = x^2, \qquad \lim_{n\to\infty} \frac{(nx-1)n(1-x)}{\binom{n}{2}} = 2(1-x)x.$$

Therefore, when n is sufficiently large, we have

$$\alpha\big(M(x,n)\big) > x^2 - \frac{\epsilon^2}{6(1+2\epsilon)}, \quad \beta\big(M(x,n)\big) > 2x(1-x) - \frac{\epsilon^2}{6(1+2\epsilon)}.$$

Now we compute $\alpha\big(M(x,n)\big) + (1+\epsilon)\beta\big(M(x,n)\big)$ given the above inequalities:

$$\begin{aligned}\alpha\big(M(x,n)\big) + (1+\epsilon)\beta\big(M(x,n)\big) &> \left(x^2 - \frac{\epsilon^2}{6(1+2\epsilon)}\right) + (1+\epsilon)\left(2x(1-x) - \frac{\epsilon^2}{6(1+2\epsilon)}\right) \\ &= x\big(x + 2(1+\epsilon)(1-x)\big) - (2+\epsilon)\frac{\epsilon^2}{6(1+2\epsilon)} \\ &> x\big(x + 2(1+\epsilon)(1-x)\big) - \frac{\epsilon^2}{2(1+2\epsilon)} \\ &= 1 + \frac{\epsilon^2}{2(1+2\epsilon)},\end{aligned}$$

where the last step follows from substituting x. This value is larger than 1 so $\theta(F) < 1+\epsilon$; since this is true for all small $\epsilon > 0$ we conclude that $\theta(F) = 1$. $\square$

Proposition 3.10. *Let F be an uncollapsible mixed graph. Then $\theta(F) = 1$.*

Proof. If there is a vertex in F that is both a head vertex and a tail vertex, then $F \not\subseteq M(x,n)$ for any choice of x and n since no vertex of $M(x,n)$ has this property. If two head vertices in F are connected by an edge, then $F \not\subseteq M(x,n)$ because there are no edges in $M(x,n)$ connecting head vertices. In either case, $\theta(F) = 1$ by Proposition 3.9.

If two tail vertices in F are connected by an edge, let $M'(x,n)$ be obtained by reversing the directions of all directed edges in $M(x,n)$, so $F \not\subseteq M'(x,n)$. Then Proposition 3.9 can be identically applied for $M'(x,n)$ and we again conclude $\theta(F) = 1$. $\square$

Lemma 3.11. *Let F be a collapsible mixed graph with at least one directed edge.Then*

$$\begin{cases} 1 + \frac{1}{\chi\left(\widetilde{F^{\triangleright}}\right)-2} \le \theta(F) \le 1 + \frac{1}{\chi\left(\widetilde{F}\right)-2}, & \textit{if } \chi\left(\widetilde{F^{\triangleright}}\right) > 2, \\ \theta(F) = \infty, & \textit{if } \chi\left(\widetilde{F^{\triangleright}}\right) = 2. \end{cases}$$

Proof. If $\chi\left(\widetilde{F^{\triangleright}}\right) = 2$, then $F^{\triangleright}$ is bipartite with exactly one directed edge, which implies F is a subgraph of $K_{\overrightarrow{t,t}}$ for some sufficiently large t, and $\theta(F) = \infty$ by Proposition 2.9.

Now assume $\chi\left(\widetilde{F^{\triangleright}}\right) > 2$. By definition of $F^{\triangleright}$, F is a subgraph of a balanced t-blowup of $F^{\triangleright}$ for some sufficiently large t. Therefore by Lemma 2.8, $\theta(F) \ge \theta\left(F^{\triangleright}[t]\right) = \theta\left(F^{\triangleright}\right)$. Since $F^{\triangleright}$ is a mixed graph with exactly one directed edge, by Theorem 1.7 $\theta\left(F^{\triangleright}\right) = 1 + \frac{1}{\chi\left(\widetilde{F^{\triangleright}}\right)-2}$. Therefore, we have that

$$\theta(F) \ge 1 + \frac{1}{\chi\left(\widetilde{F^{\triangleright}}\right) - 2}.$$

On the other hand, the underlying undirected graph $\widetilde{F}$ is a subgraph of F. Thus by Proposition 2.10,

$$\theta(F) \le \theta\left(\widetilde{F}\right) = 1 + \frac{1}{\chi\left(\widetilde{F}\right) - 2}.$$

□

The proof of Theorem 1.8 follows easily from Lemma 3.11: all that is required is to bound $\chi\left(\widetilde{F^{\triangleright}}\right)$.

Proof of Theorem 1.8. The vertex set V_0 in Definition 3.6 can be colored with no more than $\chi\left(\widetilde{F}\right)$ colors. Therefore $\chi\left(\widetilde{F^{\triangleright}}\right) \leq \chi\left(\widetilde{F}\right) + 2$. Thus by Lemma 3.11,

$$1 + \frac{1}{\chi\left(\widetilde{F}\right)} \leq \theta(F) \leq 1 + \frac{1}{\chi\left(\widetilde{F}\right) - 2}.$$

Together with Proposition 3.10, the proof is complete. □

In fact, the bound given in Theorem 1.8 is tight, since the undirected complete graph K_n attains the upper bound by Proposition 2.10, and the following (easily generalizable) example attains the lower bound:

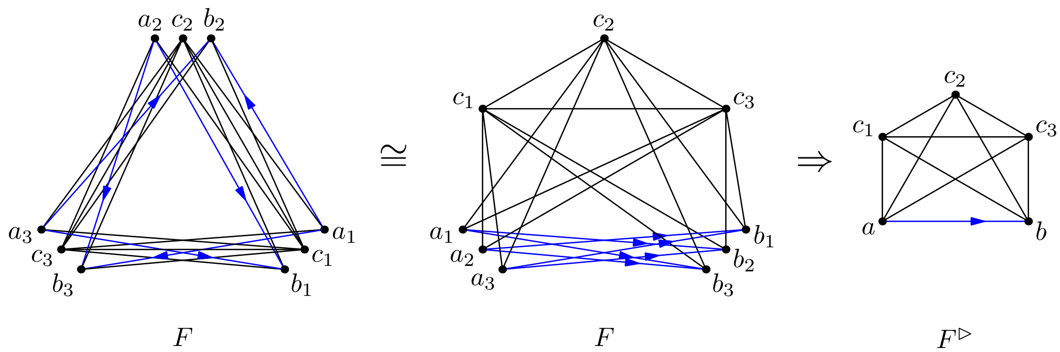


Figure 9

Example 3.12. Let F be the mixed graph shown in Figure 9. Then $\theta(F) = 1 + \frac{1}{\chi(F)} = \frac{4}{3}$.

The proof is deferred to Appendix B.

4. Mixed adjacency matrices

In this section, we consider a different extremal problem on mixed graphs, where we seek to maximize a weighted edge count over F-free mixed graphs. Formally, for positive real ρ, we define the *ρ-weighted edge count* of a mixed graph as $w_\rho(G) := e_{\mathrm{u}}(G) + \rho \cdot e_{\mathrm{d}}(G)$, and the *$\rho$-weighted degree* of a vertex $v \in V(G)$ as $\deg_\rho v = \deg_{\mathrm{u}} v + \rho \deg_{\mathrm{d}} v$—in other words, undirected edges have weight 1 and directed edges have weight ρ.

Question 4.1. For fixed $\rho \in (1, \infty)$ and mixed graph F, what is

$$\limsup_{\substack{F \nsubseteq G \\ v(G) \to \infty}} w_\rho(G) \big/ \binom{v(G)}{2},$$

the asymptotic maximum of the ρ-weighted edge density of F-free graphs?

Note that $\theta(F)$ is the largest value of ρ such that this asymptotic maximum is at most 1. This fact will later be used to develop a variational characterization of $\theta(F)$ in Section 5.

Our main result for this section, Theorem 4.42, essentially shows that it is possible to construct asymptotically maximal mixed graphs by carefully choosing a sufficiently dense smaller mixed graph and asymmetrically blowing it up. We leverage the main technique in a work of Brown, Erdős, and Simonovits [3], analyzing graph structure and edge density through an object we call a *mixed adjacency matrix*, which resemble hypergraph Lagrangians [17] or patterns [27], in form and purpose.

4.1. **Mixed adjacency matrices.** In order to precisely describe the aforementioned asymptotically maximal family we introduce the concept of a *mixed adjacency matrix*:

Definition 4.2 (Mixed adjacency matrix)**.** A *mixed adjacency matrix* A is an ordered pair of $r \times r$ matrices (U, D) such that U and D satisfy the following conditions:

(i) $U_{ij} \in \{0, 1\}$ for all $i, j \in [r]$, and U is symmetric.
(ii) $D_{ij} \in \{0, 2\}$ for all $i, j \in [r]$, and $D_{ij} \neq 0$ implies that $D_{ji} = 0$. In particular, $D_{ii} = 0$ for all $i \in [r]$.
(iii) For all $i, j \in [r]$, at most one of D_{ij} and U_{ij} is nonzero.

We say that r is the *size* of A, U is the *undirected part* of A, and D is the *directed part* of A.

A mixed adjacency matrix (U, D) of size r can be thought of as a "template" for constructing mixed graphs; the elements of U and D specify the type and direction of edge between the r parts (see Figure 10 for an illustration). The following definition makes this precise.

Definition 4.3 (Mixed matrix graphs)**.** Let $A = (U, D)$ be a mixed adjacency matrix of size r, and let $\mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{N}^r$ be a vector of nonnegative integers. Define the *mixed matrix graph* $A[\![\mathbf{x}]\!]$ as the mixed graph with vertex set given by the disjoint union $C_1 \sqcup \cdots \sqcup C_r$ (where $|C_i| = x_i$ for each $i \in [r]$), and edge set given by the following collection of vertex pairs:

(1) For each $i \in [r]$,
   - If $U_{ii} = 1$, each pair of vertices in C_i is connected by an undirected edge;
   - If $U_{ii} = 0$, no vertices of C_i are connected by any edges.

(2) For each $i, j \in [r]$ with $i \neq j$:
   - If $U_{ij} = U_{ji} = 1$, each vertex in C_i is connected to each vertex in C_j by an undirected edge;
   - If $D_{ij} = 2$ and $D_{ji} = 0$, each vertex in C_i is connected to each vertex in C_j by a directed edge with the head in C_j.
   - Otherwise ($U_{ij} = U_{ji} = D_{ij} = D_{ji} = 0$), no vertices in C_i are connected to any vertices in C_j via any type of edge.

Example 4.4. Figure 10 illustrates the mixed matrix graph $A[\![\mathbf{x}]\!]$ for $A = (U, D)$,

$$U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{x} = (2, 2, 3).$$

Note the undirected edges of $A[\![\mathbf{x}]\!]$ are specified by U, and the directed ones by D.

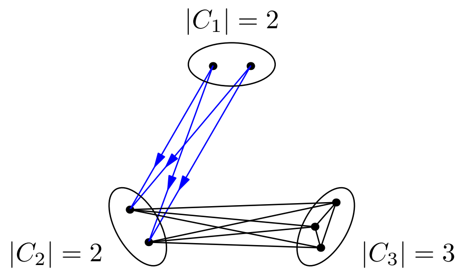


Figure 10

Definition 4.5. Let $A = (U, D)$ be a mixed adjacency matrix and $\rho \in (1, \infty)$. Its *weighted adjacency matrix* is $A_\rho := U + \rho D$ and its *symmetric part* as $A_\rho^{\mathrm{sym}} := \frac{A_\rho + A_\rho^\intercal}{2}$.

The purpose of such a representation, a simpler form of which is used in [3], is that it allows us to approximate the weighted edge count of the mixed matrix graph with a quadratic form on the weighted adjacency matrix; this is shown by the following proposition.

Proposition 4.6. *Let $A = (U, D)$ be a mixed adjacency matrix of size r, and $\rho \in (1, \infty)$. For all $\mathbf{x} \in \mathbb{N}^r$,*

$$w_\rho(A[\![\mathbf{x}]\!]) = \frac{1}{2}\mathbf{x}^\intercal A_\rho \mathbf{x} + O(\|\mathbf{x}\|_1).$$

Proof. Let $\mathbf{x} = (x_1, \ldots, x_r)$. Using notation from Definition 4.3, the counts of undirected edges and directed edges in $A[\![\mathbf{x}]\!]$ between components C_i and C_j are $\frac{1}{2}(U_{ij} + U_{ji})x_i x_j$ and $\frac{1}{2}(D_{ij} + D_{ji})x_i x_j$, respectively, and the number of undirected edges within component C_i is $U_{ii}\binom{x_i}{2} = \frac{1}{2}U_{ii}x_i^2 + O(x_i)$. Summing yields the desired result. $\square$

Unsurprisingly, for a given mixed adjacency matrix A, the maximal weighted edge density over all n-vertex mixed graphs constructed from A (i.e., all $A[\![\mathbf{x}]\!]$ for $\|\mathbf{x}\|_1 = n$) is expressible as the maximum value of a quadratic form as well.

Definition 4.7 (Density)**.** For a mixed adjacency matrix A of size r and $\rho \in (1, \infty)$, let $g_\rho(A)$ be the maximum of $\mathbf{y}^\intercal A_\rho \mathbf{y}$ over all vectors $\mathbf{y} \in \triangle^{r-1}$. (This maximum exists because $\triangle^{r-1}$ is compact). We call $g_\rho(A)$ the *density of A with respect to ρ.*

We give a crude bound on the density of any mixed adjacency matrix that will be useful later.

Proposition 4.8. *Let A be a mixed adjacency matrix and $\rho \in (1, \infty)$. Then $g_\rho(A) < \rho$.*

Proof. For all $i \neq j \in [r]$, $\left(A_\rho^{\mathrm{sym}}\right)_{ij} \in \{0, 1, \rho\}$, and for all $i \in [r]$, $\left(A_\rho^{\mathrm{sym}}\right)_{ii} \in \{0, 1\}$. Hence, letting J be the $r \times r$ matrix of all ones, and $\mathbf{y}^* \in \triangle^{r-1}$ be the vector that maximizes $\mathbf{y}^\intercal A_\rho \mathbf{y}$, we have

$$g_\rho(A) = (\mathbf{y}^*)^\intercal A_\rho \mathbf{y}^* = (\mathbf{y}^*)^\intercal \left(A_\rho^{\mathrm{sym}}\right) \mathbf{y}^* < (\mathbf{y}^*)^\intercal (\rho J) \mathbf{y}^* = \rho \|\mathbf{y}^*\|_1^2 = \rho.$$

$\square$

We would like to link the density $g_\rho(A)$ to the asymptotic maximum of $w_\rho\left(A\left[\!\left[\mathbf{x}\right]\!\right]\right)$ as $\|\mathbf{x}\|$ grows large. To do this, we will consider for each n the n-vertex graphs of the form $A\left[\!\left[\mathbf{x}\right]\!\right]$ form some x with maximal weighted edge count, and deal with their asymptotic behavior in Proposition 4.10.

Definition 4.9 (Maximal mixed matrix graph)**.** Let A be a mixed adjacency matrix of size r, choose $\rho \in (1, \infty)$, and let n be a positive integer.

(i) Let $\mathbf{x}_{\rho,A}^{(n)}$ be any vector that maximizes $w_\rho\left(A\left[\!\left[\mathbf{x}\right]\!\right]\right)$ over all vectors $\mathbf{x}$ of nonnegative integers with $\|\mathbf{x}\|_1 = n$.

(ii) Let $\mathfrak{G}_{\rho,A}^{(n)} := A\left[\!\left[\mathbf{x}_{\rho,A}^{(n)}\right]\!\right]$ be the *maximal mixed matrix graph* on n vertices.

By applying Definition 4.9 and Proposition 4.6 we can draw the following conclusion:

Proposition 4.10. *Let A be a mixed adjacency matrix and $\rho \in (1, \infty)$. For all positive integers n,*

$$w_\rho\left(\mathfrak{G}_{\rho,A}^{(n)}\right) = \frac{n^2}{2} g_\rho(A) + o\left(n^2\right).$$

We now introduce the notion of a "condensed" mixed adjacency matrix A: if it is possible to remove matching rows and columns from A without decreasing the density, then intuitively A is can be "condensed" further without affecting its density. (This is analogous to "minimal" patterns for hypergraphs; see e.g. [27]). We formalize this in terms of the principal submatrices of A:

Definition 4.11 (Principal submatrix)**.** Let $A = (U, D)$ and $A' = (U', D')$ be mixed adjacency matrices; we call A' a *principal submatrix* of A if U', D' are square matrices obtained by removing the same set of matching rows and columns from U, D, respectively. A *proper principal submatrix* of A is one which has smaller size than A.

Example 4.12. The mixed adjacency matrix $\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right)$ has three principal submatrices: itself, $([1], [0])$, and $([0], [0])$.

Definition 4.13 (Condensed mixed adjacency matrix)**.** For $\rho \in (1, \infty)$, call a mixed adjacency matrix A *condensed with respect to* ρ if $g_\rho(A') < g_\rho(A)$ for all proper principal submatrices A' of A.

When a mixed adjacency matrix is not condensed, we would like to reduce it to a condensed submatrix by removing "extra" matching rows and columns. The following makes this precise:

Proposition 4.14. *Let A be a mixed adjacency matrix, and let $\rho \in (1, \infty)$. Either A is condensed with respect to ρ, or there exists a proper principal submatrix A' of A such that A' is condensed with respect to ρ, and $g_\rho(A') = g_\rho(A)$.*

Proof. Take A' to be a principal submatrix of A of smallest size such that $g_\rho(A') = g_\rho(A)$. $\square$

We thus introduce the notation $A \xrightarrow{\text{sub}} A'$ to denote the relation that A' is a condensed principal submatrix of A and $g_\rho(A') = g_\rho(A)$; ρ will be clear from context whenever this is used.

4.2. **Convergence.** Considering mixed adjacency matrices as templates for asymmetrical blowups, we introduce in this section a highly useful "convergence lemma," which shows that the parts in the maximal asymmetric blowup of a condensed matrix must be approximately equal to some fixed ratio (with all parts nonzero). This will be useful later on, because it allows us to describe maximal blowups by a unique *optimal vector*.

The proofs in this section are mostly standard sequence analysis, with techniques such as Zykov symmetrization borrowed from hypergraph Lagrangian methods; this is similar to Claim 2.(C) in [3] (though more complex).

First, we give a lemma, again using the method of Zykov in [1], which shows that all vertices in a maximal mixed graph have approximately the same weighted degree up to some constant difference.

Lemma 4.15 (Zykov symmetrization)**.** *Let A be a mixed adjacency matrix, n be a positive integer, and $\rho \in (1, \infty)$. Then $\left|\deg_\rho v_1 - \deg_\rho v_2\right| \le \rho$ for any vertices v_1, v_2 of $\mathfrak{G}^{(n)}_{\rho,A}$.*

Proof. Let $A = (U, D)$, and $\mathbf{x}^{(n)}_{\rho,A}$ be the vector from Definition 4.9. By construction, if v_1, v_2 are in the same component C_i then $\deg_\rho v_1 = \deg_\rho v_2$. Otherwise, assume without loss of generality that $v_1 \in C_1$ and $v_2 \in C_2$, and $\deg_\rho v_1 > \deg_\rho v_2 + \rho$.

Consider the vector $\mathbf{x}' = \mathbf{x}^{(n)}_{\rho,A} + (1, -1, 0, \ldots, 0)$ (equivalent to moving the vertex v_2 from C_2 to C_1). We have $\mathbf{x}' \in \mathbb{N}^r$ and $\|\mathbf{x}'\|_1 = n$, but

$$w_\rho\left(A\left[\!\left[\mathbf{x}'\right]\!\right]\right) = w_\rho\left(A\left[\!\left[\mathbf{x}^{(n)}_{\rho,A}\right]\!\right]\right) + \deg_\rho v_1 - \deg_\rho v_2 - \left(A^{\mathrm{sym}}_\rho\right)_{12} + U_{11} > w_\rho\left(A\left[\!\left[\mathbf{x}^{(n)}_{\rho,A}\right]\!\right]\right),$$

where the last inequality follows from $\deg_\rho v_1 - \deg_\rho v_2 > \rho \ge \left(A^{\mathrm{sym}}_\rho\right)_{12}$ and $U_{11} \ge 0$. This contradicts the maximality of $\mathbf{x}^{(n)}_{\rho,A}$. $\square$

Now we are ready to state, and prove, the convergence lemma.

Lemma 4.16 (Convergence)**.** *For $\rho \in (1, \infty)$ and a condensed mixed adjacency matrix A of size r, we have*

(i) $\lim_{n\to\infty} \frac{1}{n}\mathbf{x}_{\rho,A}^{(n)} = \mathbf{y}^*$ *exists (regardless of the choice of* $\mathbf{x}_{\rho,A}^{(n)}$*).*

(ii) $\mathbf{y}^*$ *is the unique solution with nonnegative coordinates to*

$$\mathbf{y} \in \triangle^{r-1}, \quad (A_\rho^{\mathrm{sym}})\mathbf{y} = g_\rho(A)\mathbf{1}, \tag{4.1}$$

(iii) all coordinates of $\mathbf{y}^*$ *are strictly positive;*

(iv) $\mathbf{y}^*$ *is the unique vector that maximizes* $\mathbf{y}^\intercal A_\rho \mathbf{y}$ *among all vectors* $\mathbf{y} \in \triangle^{r-1}$*.*

Proof. We break the proof into three claims (proofs deferred to Appendix C): the first establishes a relation between (4.1) and the argmax of $\mathbf{y}^\intercal A_\rho \mathbf{y}$, the second shows that $\lim_{n\to\infty} \frac{1}{n}\mathbf{x}_{\rho,A}^{(n)}$ exists, and the third shows that the limit has the desired properties.

Claim 4.17. Any vector $\mathbf{y} \in \triangle^{r-1}$ which maximizes $\mathbf{y}^\intercal A_\rho \mathbf{y}$ satisfies (4.1).

Claim 4.18. If (4.1) has a solution, then it is unique and has all positive coordinates.

Now we deal with the sequence $\frac{1}{n}\mathbf{x}_{\rho,A}^{(n)}$. Because $\frac{1}{n}\mathbf{x}_{\rho,A}^{(n)} \in [0,1]^r$ for all n, the sequence has limit points. Suppose $\hat{\mathbf{y}} = (y_1, \dots, y_r)$ is a limit point; then, it is clear that $\hat{\mathbf{y}} \in \triangle^{r-1}$.

Claim 4.19. The vector $\hat{\mathbf{y}}$ is a solution of (4.1).

Combining the three claims gives the desired conclusion: the limit $\frac{1}{n}\mathbf{x}_{\rho,A}^{(n)}$ exists; it is the unique solution to (4.1); it has all positive coefficients; and it maximizes $\mathbf{y}^\intercal A_\rho \mathbf{y}$. □

We give $\mathbf{y}^*$ in Lemma 4.16 its own name:

Definition 4.20 (Optimal vector)**.** Let $\rho \in (1, \infty)$ and A be a condensed mixed adjacency matrix with respect to ρ. We will denote the unique solution to (4.1) by $\mathbf{y}_{\rho,A}^*$, the *optimal vector* of A with respect to ρ.

A simple corollary of Lemma 4.16 is that the part sizes of $\mathfrak{G}_{\rho,A}^{(n)}$ become arbitarily large, since each term in $\mathbf{y}_{\rho,A}^*$ is positive.

Corollary 4.21. *Let $\rho \in (1, \infty)$ and A be a condensed mixed adjacency matrix with respect to ρ. For any positive integer t there exists N such that $A[\![t\mathbf{1}]\!] \subseteq \mathfrak{G}_{\rho,A}^{(n)}$ for all $n \geq N$.*

Another extremely useful implication is the following, which strengthens the Zykov symmetrization claim from before:

Proposition 4.22 (Lagrangians)**.** *Let $\rho \in (1, \infty)$ and $A = (U, D)$ be a condensed mixed adjacency matrix with respect to ρ. For any $i \neq j$, if $U_{ii} = U_{jj}$ then $\left(A_\rho^{\mathrm{sym}}\right)_{ij} > U_{ii}$. (Hence if all U_{ii} are equal, then $A[\![\mathbf{1}]\!]$ is* complete*, meaning every pair of vertices is connected by some edge.)*

Proof. This is analogous to the proof of Lemma 1 in [3]. Let $\mathbf{y}^*_{\rho,A} = (y_1, \dots, y_r)$; without loss of generality assume $i = 1$, $j = 2$, and $\sum_{k=3}^r \left(A_\rho^{\mathrm{sym}}\right)_{1k} y_k \geq \sum_{k=3}^r \left(A_\rho^{\mathrm{sym}}\right)_{2k} y_k$. Let $\mathbf{y}' = (y_1 + y_2, 0, y_3, \dots, y_r)$. By Definition 4.20 $\mathbf{y}^*_{\rho,A}$ is the unique optimal vector, so indeed

$$\begin{aligned}
0 &< \left(\mathbf{y}^*_{\rho,A}\right)^\intercal A_\rho \mathbf{y}^*_{\rho,A} - (\mathbf{y}')^\intercal A \mathbf{y}' \\
&= y_2^2 \left(U_{22} - U_{11}\right) + y_2 \sum_{k=3}^r 2\left(\left(A_\rho^{\mathrm{sym}}\right)_{2k} - \left(A_\rho^{\mathrm{sym}}\right)_{1k}\right) y_k + 2\big((A_\rho^{\mathrm{sym}})_{12} - U_{11}\big) y_1 y_2 \\
&\leq 2\big((A_\rho^{\mathrm{sym}})_{12} - U_{11}\big) y_1 y_2.
\end{aligned}$$

□

4.3. **Augmentation of mixed adjacency matrices.** The work in this section is a more complex version of Section 3 in [3]. The main idea of the section is that, in order to find a small mixed adjacency matrix that yields a ρ-weighted blowup of high density, we may grow a template adjacency matrix by systematically adding vertices while guaranteeing that the density increases at each step. (In later sections we will analyze the procedure given here to ensure that it terminates after finitely many steps at an "optimal" mixed graph to blow up.)

The key concept of *augmentation* is to begin with a condensed matrix A of size r, then add a row and column to A (equivalent to forming a new vertex part C_{r+1}) to form a new matrix A' of size $r + 1$ and higher density. The key result, Lemma 4.25, will enable the finding of an asymptotically maximal mixed adjacency matrix by repeated augmentation in later sections.

Definition 4.23 (Augmentation)**.** Fix $\rho \in (1, \infty)$. Let $B = (U, D)$ be a mixed adjacency matrix of size $r + 1$ with $U_{(r+1)(r+1)} = 0$. Let $A = (U', D')$ be the principal submatrix of B obtained by removing the $(r + 1)$th row and $(r + 1)$th column from both U and D. Further suppose A is condensed with respect to ρ, with optimal vector $\mathbf{y}^*_{\rho,A} = (y_1, \dots, y_r)$, and

$$\sum_{j=1}^r \left(A_\rho^{\mathrm{sym}}\right)_{(r+1)j} y_j > g_\rho(A). \tag{4.2}$$

Then we say that B is obtained from A by *augmentation*, denoted by $A \xrightarrow{\mathrm{aug}} B$.

As alluded to earlier, the purpose of (4.2) is to ensure that the augmented matrix has higher density than the original.

Proposition 4.24. *If $A \xrightarrow{\mathrm{aug}} B$ then $g_\rho(A) < g_\rho(B)$.*

Proof. This is algebraic manipulation identical to that in Lemma 2 of [3]. □

Now we are ready to state and prove the augmentation lemma (this corresponds to Lemma 4 in [3]). Roughly speaking, it gives a useful condition for when a mixed adjacency matrix, can be augmented while still having a large blowup as a subgraph of some large mixed graph G. This result will be key in Proposition 4.31 and Proposition 4.34, where we use it to repeatedly augment a mixed adjacency matrix (thus increasing its density), while ensuring that blowups of the matrix are F-free by keeping them as subgraphs of a large F-free mixed graph G.

Lemma 4.25 (Augmentation lemma). *Let $\rho \in (1, \infty)$, A be a condensed mixed adjacency matrix with respect to ρ, and ϵ be a positive real. For all positive integers m, there exists a positive integer $N = N(A, \rho, \epsilon, m)$ such that: for any mixed graph G on n vertices (n sufficiently large) that satisfies*

(i) $\deg_\rho v \geq (g_\rho(A) + \epsilon)n$ *for all vertices* $v \in V(G)$,

(ii) $\mathfrak{G}^{(N)}_{\rho,A} \subseteq G$,

there exists a mixed adjacency matrices B such that $A \xrightarrow{\text{aug}} B$, and $\mathfrak{G}^{(m)}_{\rho,B} \subseteq G$.

The proof will be deferred to Appendix D.

4.4. **Forbidding a subgraph.** In this section, we introduce the forbidden graph F for the first time, and consider mixed adjacency matrices which do not contain F in any of their blouwps. We construct a set of template adjacency matrices which we will later iteratively augment into the asymptotically maximal blowup mixed graph (in fact, this set will consist only of mixed adjacency matrices of size 1).

This, along with the following section, bear some resemblance to the proof of Theorem 1 in [3], with slightly different conclusions and significantly increased complexity due to the existence of undirected edges. We first define the notion of *containment*: roughly speaking, the mixed adjacency matrix A is contained in a sequence of mixed graphs (G_k) if every mixed graph of the form $A[\![\mathbf{x}]\!]$ is a subgraph of some $G \in (G_k)$.

Definition 4.26 (Containment). For $\rho \in (1, \infty)$, let A be a condensed mixed adjacency matrix with respect to ρ, and let $\left(G^{(n)}\right)_{n=1}^{\infty}$ be a sequence of mixed graphs. We say A is *contained* in the sequence if for any positive integer m, there exists integer n such that $\mathfrak{G}^{(m)}_{\rho,A} \subseteq G^{(n)}$.

It is easy to see by Corollary 4.21 that the following three conditions are equivalent:

(i) A is contained in the sequence $\left(G^{(n)}\right)_{n=1}^{\infty}$;

(ii) there is an infinite sequence of integers $m_1 < m_2 < \cdots$ such that for each m_i in the sequence, there exists integer n so that $\mathfrak{G}^{(m_i)}_{\rho,A} \subseteq G^{(n)}$.

(iii) for any positive integer t, there exists integer n such that $A[\![t\mathbf{1}]\!] \subseteq G^{(n)}$;

Now, following the proof of Theorem 1 in [3], we briefly discuss the Zarankiewicz problem [37] corresponding to Question 4.1; the extremal mixed graphs that arise will in fact be the large mixed graphs G on which we later apply the augmentation lemma (Lemma 4.25).

Definition 4.27. Let $\rho \in (1, \infty)$, F be a mixed graph, and n be a positive integer. Define $d^{(n)}_{\rho,F}$ as the maximum possible minimum ρ-weighted degree among all F-free mixed graphs G on n vertices, and $Z^{(n)}_{\rho,F}$ as a mixed graph on n which attains this maximum.

$$d^{(n)}_{\rho,F} = \max_{\substack{v(G)=n \\ F \not\subseteq G}} \left\{ \min_{v \in V(G)} \deg_\rho v \right\} \quad , \quad Z^{(n)}_{\rho(F)} = \arg\max_{\substack{v(G)=n \\ F \not\subseteq G}} \left\{ \min_{v \in V(G)} \deg_\rho v \right\}.$$

Define $a^*_{\rho,F} = \limsup_{n\to\infty} \frac{d^{(n)}_{\rho,F}}{n}$. Then there exists an infinite sequence of increasing integers $N_{\rho,F} = (n_1, n_2, \dots)$ such that

$$\lim_{j\to\infty} \frac{d^{(n_j)}_{\rho,F}}{n_j} = a^*_{\rho,F}.$$

The mixed graph $Z^{(n)}_{\rho,F}$ is specifically constructed for use as the large subgraph G in Lemma 4.25: all of its vertices have high minimum degree (equal to $d^{(n)}_{\rho,F}$, which we will bound later), and $Z^{(n)}_{\rho,F}$ is F-free, to ensure that blowups of the augmented matrices remain F-free.

Definition 4.28. For mixed adjacency matrix A and mixed graph F, we say that A is F-free if the sequence $(A\,[\![t\mathbf{1}]\!])_{t=1}^{\infty}$ is F-free.

Denote the two mixed graphs of size 1 by $\mathbf{0} = ([0],[0])$ and $K = ([1],[0])$. The latter is named thus because $K\,[\![t\mathbf{1}]\!]$ is the complete graph K_t.

Definition 4.29. Let F be a mixed graph and $\rho \in (1,\infty)$. Define $\mathcal{E}_{\rho,F}$ as the union of $\{\mathbf{0}\}$, and the set of all condensed mixed adjacency matrices with diagonal elements of their undirected parts all equal to 1 which are contained in the sequence $\left(Z^{(n)}_{\rho,F}\right)_{n\in N_{\rho,F}}$.

Proposition 4.30. *Let F be a mixed graph with $\theta(F) \in (1,\infty)$. For any $\rho \in (1,\infty)$, either $\mathcal{E}_{\rho,F} = \{\mathbf{0}\}$ or $\mathcal{E}_{\rho,F} = \{\mathbf{0}, K\}$.*

Proof. Since $\mathfrak{G}^{(m)}_{\rho,\mathbf{0}}$ is simply an empty graph on m vertices, it is clear that $\mathbf{0} \in \mathcal{E}_{\rho,F}$ for all ρ and F. Thus it suffices to show that any condensed mixed adjacency matrix $A = (U, D)$ of size at least 2 with all diagonal elements of U equal to 1 cannot be in $\mathcal{E}_{\rho,F}$. By Proposition 4.22, $\{D_{12}, D_{21}\} = \{2, 0\}$; without loss of generality suppose $D_{12} = 2$. For any positive integer t, there exists $n \in N_{\rho,F}$ such that $A\,[\![t\mathbf{1}]\!] \subseteq Z^{(n)}_{\rho,F}$. We have $F \not\subseteq A[t\mathbf{1}]$ since $Z^{(n)}_{\rho,F}$ is F-free by Definition 4.27. Now consider the mixed adjacency matrix

$$A' = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right).$$

We have $A'\,[\![t\mathbf{1}]\!] \subseteq A\,[\![t\mathbf{1}]\!]$, so $F \not\subseteq A'\,[\![t\mathbf{1}]\!]$ for all positive integers t, which means $F \not\subseteq M(x, s)$ for any positive integer s and any $x \in (0, 1)$. This implies $\theta(F) = 1$ by Proposition 3.9, contradiction. ∎

4.5. **Extremal mixed adjacency matrices.** We are finally ready to resolve Question 4.1 by the process previously alluded to. We begin with the mixed adjacency matrices of size 1 in $\mathcal{E}_{\rho,F}$, and consider all matrices which can be obtained from them by a series of augmentations. We first certify that there are only finitely many such matrices, then choose the one with highest density. We show that the mixed graphs obtained by blowing it up optimally are in fact asymptotically extremal F-free mixed graphs. Finally, we establish a finite set of mixed adjacency matrices $\mathcal{M}_F$ which depends solely on F and not ρ which is guaranteed to contain the optimal matrix, in preparation for the next section, where we will give the variational characterization of $\theta(F)$ by varying ρ for fixed F.

In our first step, we deal with the two cases where $K \in \mathcal{E}_{\rho,F}$ and $K \notin \mathcal{E}_{\rho,F}$ separately. The approaches in both cases are similar—for any matrix in the sequence, we apply Lemma 4.25 using the F-free supergraph $Z^{(n)}_{\rho,F}$ to attempt to find an augmentation of the matrix which is still F-free. In either case, we obtain an equation for $\lim_{n\to\infty} \frac{1}{n} d^{(n)}_{\rho,F}$. We do this for the $k \in \mathcal{E}_{\rho,F}$ case first, in a single proposition:

Proposition 4.31. *Let F be a mixed graph with $\theta(F) \in (1,\infty)$. For any $\rho \in (1,\infty)$, if $K \in \mathcal{E}_{\rho,F}$ then the sequence $\frac{d^{(n)}_{\rho,F}}{n}$ converges and*

$$\lim_{n\to\infty} \frac{d^{(n)}_{\rho,F}}{n} = 1 = g_\rho(K).$$

Proof. Since K is contained in the sequence $\left(Z^{(n)}_{\rho,F}\right)_{n\in N_{\rho,F}}$, for any positive integer t there exists $n \in N_{\rho,F}$ such that the undirected complete graph $K_t = K\,[\![t\mathbf{1}]\!] \subseteq Z^{(n)}_{\rho,F}$, thus $F \not\subseteq K_t$ for all t, implying $d^{(t)}_{\rho,F} \geq t-1$ for all positive integers t. Hence

$$\liminf_{n\to\infty} \frac{d^{(n)}_{\rho,F}}{n} \geq \liminf_{n\to\infty} \frac{n-1}{n} = 1.$$

If $\limsup_{n\to\infty} \frac{d^{(n)}_{\rho,F}}{n} > 1$, then there exists $\epsilon > 0$ such that $d^{(n)}_{\rho,F} > (1+\epsilon)n$ for all sufficiently large $n \in N_{\rho,F}$. All conditions for Lemma 4.25 on K and $Z^{(n)}_{\rho,F}$ are met for large n, so there exist mixed adjacency matrix B such that $K \xrightarrow{\text{aug}} B$, and $\mathfrak{G}^{(m)}_{\rho,B} \subseteq Z^{(n)}_{\rho,F}$. This implies $F \not\subseteq \mathfrak{G}^{(m)}_{\rho,B}$. Since augmentation increases density, there are only two possible B values:

$$B = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 2\\0 & 0\end{bmatrix}\right) \text{ or } \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\2 & 0\end{bmatrix}\right).$$

So by the same reasoning as in the proof of Proposition 4.30, we must have $F \not\subseteq M(x,t)$ for any positive integer t and real number $x \in (0,1)$, thus $\theta(F) = 1$, contradiction. Therefore $\limsup_{n\to\infty} \frac{d^{(n)}_{\rho,F}}{n} \leq 1$.

Hence we conclude that $\lim_{n\to\infty} \frac{d^{(n)}_{\rho,F}}{n} = 1 = g_\rho(K)$. ∎

The case where $K \notin \mathcal{E}_{\rho,F}$ is more complicated, since the set of mixed adjacency matrices obtainable from the sequence can now be quite large.

Definition 4.32. Let F be a mixed graph with $\theta(F) \in (1,\infty)$. Given $\rho \in (1,\infty)$, suppose $K \notin \mathcal{E}_{\rho,F}$. Define $\mathcal{B}_{\rho,F}$ as the set of all mixed adjacency matrices B which are condensed with respect to ρ, and such that B is contained in the sequence of mixed graphs $\left(Z_{\rho,F}^{(n)}\right)_{n\in N_{\rho,F}}$, and B is obtainable from some finite (possibly empty) sequence

$$\mathbf{0} \xrightarrow{\text{aug}} B_1 \xrightarrow{\text{sub}} B_2 \xrightarrow{\text{aug}} B_3 \xrightarrow{\text{sub}} \cdots \xrightarrow{\text{sub}} B_{2k} = B.$$

We would like to choose the mixed adjacency matrix B of maximal density from $\mathcal{B}_{\rho,F}$. The following proposition ensures that this is in fact possible:

Proposition 4.33 (Finiteness)**.** *If $K \notin \mathcal{E}_{\rho,F}$, then the set $\mathcal{B}_{\rho,F}$ is finite.*

Proof. Suppose for the sake of contradiction that $\mathcal{B}_{\rho,F}$ is infinite. Note that the undirected part of each $B \in \mathcal{B}_{\rho,F}$ has all diagonal elements equal to zero, since augmentation only adds 0s to the diagonals.

Because there are only finitely many mixed adjacency matrices of any given size, there are arbitrarily large mixed adjacency matrices $B \in \mathcal{B}_{\rho,F}$ contained in the sequence $\left(Z_{\rho,F}^{(n)}\right)_{n\in N_{\rho,F}}$.

Take $B \in \mathcal{B}_{\rho,F}$ of arbitrarily large size. Let $B' = (U', D')$ be the mixed adjacency matrix of the same size, with U' having all 0s on the diagonal and all 1s elsewhere, and D' having all elements 0. By Proposition 4.22, each pair of vertices in $B\,[\![\mathbf{1}]\!]$ are connected by an edge, so $B'\,[\![\mathbf{1}]\!] \subseteq B\,[\![\mathbf{1}]\!]$. Hence the fact that B is contained in $\left(Z_{\rho,F}^{(n)}\right)_{n\in N_{\rho,F}}$ implies that B' is as well, which means we can find arbitrarily large complete undirected graphs contained in $\left(Z_{\rho,F}^{(n)}\right)_{n\in N_{\rho,F}}$. Therefore K is contained in $\left(Z_{\rho,F}^{(n)}\right)_{n\in N_{\rho,F}}$, implying $K \in \mathcal{E}_{\rho,F}$, contradiction. □

Now we give the aforementioned result on $\lim_{n\to\infty} \frac{1}{n} d_{\rho,F}^{(n)}$.

Proposition 4.34. *Let F be a mixed graph with $\theta(F) \in (1,\infty)$, and $\rho \in (1,\infty)$, assume $K \notin \mathcal{E}_{\rho,F}$, then the sequence $\frac{d_{\rho,F}^{(n)}}{n}$ converges and*

$$\lim_{n\to\infty} \frac{d_{\rho,F}^{(n)}}{n} = \max_{B\in\mathcal{B}_{\rho,F}} \left\{g_\rho(B)\right\}.$$

Proof. Let B^* be any mixed adjacency matrix in $\mathcal{B}_{\rho,F}$ with maximal density. By Proposition 4.10 and Lemma 4.15, given any positive integer n, all vertices v in $\mathfrak{G}_{\rho,B^*}^{(n)}$ have weighted degree $\deg_\rho v = g_\rho\left(B^*\right) n + o(n)$. Since $F \not\subseteq \mathfrak{G}_{\rho,B^*}^{(n)}$, we have $d_{\rho,F}^{(n)} \geq g\left(B^*\right) n + o(n)$ for all n, and

$$\liminf_{n\to\infty} \frac{d_{\rho,F}^{(n)}}{n} \geq g_\rho\left(B^*\right).$$

On the other hand, suppose

$$\limsup_{n\to\infty} \frac{d_{\rho,F}^{(n)}}{n} = \lim_{\substack{n\in N_{\rho,F}\\ n\to\infty}} \frac{d_{\rho,F}^{(n)}}{n} > g_\rho\left(B^*\right).$$

There exists $\epsilon > 0$ such that any sufficiently large $n \in N_{\rho,F}$ satisfies $d_{\rho,F}^{(n)} > \left(g_\rho\left(B^*\right) + \epsilon\right) n$. There exists some $n \in N_{\rho,F}$ such that $\mathfrak{G}_{\rho,B^*}^{(N)} \subseteq Z_{\rho,F}^{(n)}$. We established earlier that $\deg_\rho v > \left(g_\rho\left(B^*\right) + \epsilon\right) n$ for all vertices v of $Z_{\rho,F}^{(n)}$. Hence Lemma 4.25 on B^* and $Z_{\rho,F}^{(n)}$ for large n,states that there exist mixed adjacency matrix B' such that $B^* \xrightarrow{\text{aug}} B'$ (dependent on m and $n = n(m)$) and $\mathfrak{G}_{\rho,B'}^{(m)} \subseteq Z_{\rho,F}^{(n)}$. We then take $B' \xrightarrow{\text{sub}} B''$ then B'' is condensed and $\mathfrak{G}_{\rho,B''}^{(m)} \subseteq Z_{\rho,F}^{(n)}$.

Note there are only finitely many such matrices B'' because the size of each is at most one more than the size of B^*. Thus there is at least one such matrix B'' such that $\mathfrak{G}_{\rho,B''}^{(m)} \subseteq Z_{\rho,F}^{(n)}$ is true for infinitely many pairs $m, n(m)$. This B'' is contained in the sequence $Z_{\rho,F}^{(n)}$, so $B'' \in \mathcal{B}_{\rho,F}$ and $g_\rho(B'') > g_\rho\left(B^*\right)$, contradicting the maximality of $g_\rho\left(B^*\right)$ in $\mathcal{B}_{\rho,F}$. Therefore

$$\limsup_{n\to\infty} \frac{d_{\rho,F}^{(n)}}{n} \le g_\rho(B^*).$$

$\square$

Combining the two cases, Proposition 4.31 and Proposition 4.34 allow for the following definition and corollary:

Definition 4.35 (Extremal mixed adjacency matrix)**.** Let F be a mixed graph such that $\theta(F) \in (1, \infty)$, and let $\rho \in (1, \infty)$. We use $B_{\rho,F}^*$ to denote the *extremal mixed adjacency matrix* for F with respect to ρ, defined by the following:

(i) $B_{\rho,F}^* \coloneqq K$ if $K \in \mathcal{E}_{\rho,F}$;
(ii) $B_{\rho,F}^* \coloneqq \arg\max_{B\in\mathcal{B}_{\rho,F}}\ g_\rho(B)$ otherwise.

Corollary 4.36. *For a mixed graph F with $\theta(F) \in (1, \infty)$, and for $\rho \in (1, \infty)$, the sequence $\frac{d_{\rho,F}^{(n)}}{n}$ converges to the density of its extremal mixed adjacency matrix, namely*

$$\lim_{n\to\infty} \frac{d_{\rho,F}^{(n)}}{n} = g_\rho\left(B_{\rho,F}^*\right).$$

Furthermore, $B_{\rho,F}^*$ is contained in the sequence of mixed graphs $\left(Z_{\rho,F}^{(n)}\right)_{n\in N_{\rho,F}}$ and therefore $F \not\subseteq \mathfrak{G}_{\rho,B_{\rho,F}^*}^{(n)}$ for any positive integer n.

Our aim is now to show that the maximal graphs obtained by optimally blowing up $B_{\rho,F}^*$ are asymptotically extremal. We first make this notion precise in Definition 4.37, then prove it in Lemma 4.38.

Definition 4.37 (Asymptotically extremal sequence)**.** Let F be a mixed graph; we say that the sequence of mixed graphs $\left(G^{(n)}\right)_{n=1}^{\infty}$, where each $G^{(n)}$ has n vertices, is *asymptotically extremal* for F (with respect to ρ) if

- $F \not\subseteq G^{(n)}$ for any n;
- for any $\epsilon > 0$, for sufficiently large n, any n-vertex mixed graph G such that $w_\rho(G) > w_\rho\left(G^{(n)}\right) + \epsilon n^2$ must satisfy $F \subseteq G$.

Lemma 4.38 (Extremal lemma)**.** *Let F be a mixed graph with $\theta(F) \in (1,\infty)$, and $\rho \in (1,\infty)$. The sequence of maximal mixed graphs $\left(\mathfrak{G}^{(n)}_{\rho,B^*_{\rho,F}}\right)_{n=1}^{\infty}$ is asymptotically extremal for F.*

Proof. We use B^* to denote $B^*_{\rho,F}$ in this proof. Let ϵ be a positive real and G be an n-vertex mixed graph (for large n) such that $w_\rho(G) > w_\rho\left(\mathfrak{G}^{(n)}_{\rho,B^*}\right) + \epsilon n^2$. By Proposition 4.10, when n is sufficiently large, $w_\rho\left(\mathfrak{G}^{(n)}_{\rho,B^*}\right) > \left(g_\rho(B^*) - \frac{\epsilon}{2}\right)\frac{n^2}{2}$, so $w_\rho(G) > \left(g_\rho(B^*) + \frac{3\epsilon}{2}\right)\frac{n^2}{2}$.

Claim 4.39. When n is sufficiently large, there exists a subgraph $H \subseteq G$ on k vertices where $\sqrt{\frac{\epsilon}{2\rho}}n \le k \le n$, and

$$\min_{v\in V(H)} \deg_\rho v > \left(g_\rho(B^*) + \frac{\epsilon}{2}\right)k.$$

(Here $\deg_\rho v$ refers to the degree of v in H.)

Proof. Construct a sequence of mixed graphs $G^{(n)} = G, G^{(n-1)}, \ldots$, where $G^{(n-j)}$ is obtained from $G^{(n-j+1)}$ by removing a vertex v with $\deg_\rho v \le \left(g_\rho(B^*) + \frac{\epsilon}{2}\right)(n-j+1)$. The sequence terminates when such a vertex does not exist. Then when n is sufficiently large, $G^{(n-j)}$ in the sequence has

$$\begin{aligned}
w_\rho\left(G^{(n-j)}\right) &> \left(g_\rho(B^*) + \frac{3\epsilon}{2}\right)\frac{n^2}{2} - \left(g_\rho(B^*) + \frac{\epsilon}{2}\right)\frac{j(2n-j+1)}{2} \\
&\ge \left(g_\rho(B^*) + \frac{\epsilon}{2}\right)\frac{(n-j)^2}{2} + \frac{1}{2}\epsilon n^2 + O(n) \\
&\ge \frac{1}{4}\epsilon n^2.
\end{aligned}$$

But $w_\rho\left(G^{(n-j)}\right) \le \rho\binom{n-j}{2}$, so $\rho\frac{(n-j)^2}{2} > \frac{1}{4}\epsilon n^2$, which implies $n-j \ge \sqrt{\frac{\epsilon}{2\rho}}n$, i.e. the sequence of mixed graphs must terminate at some graph H on $k \ge \sqrt{\frac{\epsilon}{2\rho}}n$ vertices. Then, by construction,

$$\min_{v\in V(H)} \deg_\rho v > \left(g_\rho(B^*) + \frac{\epsilon}{2}\right)k.$$

□

Now we can finish the proof of the lemma. For the sake of contradiction, assume there exists $\epsilon > 0$, an infinite sequence of integers $n_1 < n_2 < \cdots$, and mixed graphs G_j on n_j

vertices, such that for all sufficiently large j, $w_\rho(G_j) > w_\rho\left(\mathfrak{G}^{(n_j)}_{\rho,B^*}\right) + \epsilon n_j^2$ and $F \not\subseteq G_j$. By Claim 4.39, there is a subgraph $H_j \subseteq G_j$ (therefore $F \not\subseteq H_j$) on k_j vertices with $\sqrt{\frac{\epsilon}{2\rho}} n_j \leq k_j \leq n_j$ and $\min_{v \in V(H_j)} \deg_\rho v > \left(g_\rho(B^*) + \frac{\epsilon}{2}\right) k_j$. Therefore $\frac{d^{(k_j)}_{\rho,F}}{k_j} > g_\rho(B^*) + \frac{\epsilon}{2}$ for all sufficiently large j (recall Definition 4.27). Thus

$$\limsup_{n\to\infty} \frac{d^{(n)}_{\rho,F}}{n} = \lim_{j\to\infty} \frac{d^{k_j}_{\rho,F}}{k_j} \geq g_\rho(B^*) + \frac{\epsilon}{2}.$$

But this contradicts Corollary 4.36 which states that $\lim_{n\to\infty} \frac{d^{(n)}_{\rho,F}}{n} = g_\rho(B^*)$. □

Finally, note that while the set $\mathcal{B}_{\rho,F}$ is finite, its size varies based on the value of ρ. To conclude this section we construct the set $\mathcal{M}_F$ which is a finite set guaranteed to contain $B^*_{\rho,F}$ that depends only on the forbidden graph F.

Definition 4.40. Let F be a mixed graph with $\theta(F) \in (1,\infty)$, define $\mathcal{M}_F$ as the union of $\{K\}$ and the set of mixed adjacency matrices $A = (U, D)$ that satisfy the following conditions:

(i) A is F-free;
(ii) the size of A is at most $\chi\left(\widetilde{F^\triangleright}\right) - 1$ (note $F^\triangleright$ is defined since F is collapsible by $\theta(F) > 1$).
(iii) all diagonal entries $U_{ii} = 0$;
(iv) D is not the zero matrix;
(v) and $U_{ij} + U_{ji} + D_{ij} + D_{ji} > 0$ for all $i \neq j$.

Indeed, the size of $\mathcal{M}_F$ is finite: There are at most $3^{(r-1)r/2}$ elements of $\mathcal{M}_F$ of size r, since for each $1 \leq i < j \leq r$ we have $(U_{ij}, U_{ji}, D_{ij}, D_{ji})$ either $(1,1,0,0)$, $(0,0,2,0)$, or $(0,0,0,2)$.

Proposition 4.41. *Let F be a mixed graph with at least one directed edge and $\theta(F) \in (1,\infty)$. Then $B^*_{\rho,F} \in \mathcal{M}_F$ for any $\rho \in (1,\infty)$.*

Proof. If $K \in \mathcal{E}_{\rho,F}$, then $B^*_{\rho,F} = K \in \mathcal{M}_F$ by Definition 4.35. Otherwise $K \notin \mathcal{E}_{\rho,F}$. Let $B^*_{\rho,F} = (U, D)$.

Conditions (i) and (iii) are met by Definition 4.32, and (v) is met by Proposition 4.22. Note that if D is the zero matrix then $g(B^*_{\rho,F}) = \max_{\mathbf{y}\in\triangle^{r-1}} \mathbf{y}^\intercal U \mathbf{y} < 1$, contradiction, so (iv) is true. By (iv) we know $B^*_{\rho,F}[\![\mathbf{1}]\!]$ is a complete graph with at least one directed edge. Hence its size is at most $\chi\left(\widetilde{F^\triangleright}\right) - 1$, since otherwise for large enough t' and t, we would have $F \subseteq F^\triangleright[t] \subseteq B^*_{\rho,F}[\![t'\mathbf{1}]\!]$, contradiction. This means (ii) is true as well, so all conditions are satisfied and $B^*_{\rho,F} \in \mathcal{M}_F$. □

The following theorem concludes the section; it gives the existence of an "asymptotically extremal mixed adjacency matrix," a matrix whose sequence of maximal graphs is F-free and asymptotically extremal, and guarantees that the matrix is contained in the finite predetermined set $\mathcal{M}_F$.

Theorem 4.42. *Let F be a mixed graph with at least one directed edge and $\theta(F) \in (1,\infty)$, and let $\rho \in (1,\infty)$ be a fixed constant. Then,*

$$\limsup_{\substack{v(G)\to\infty \\ F\not\subseteq G}} w_\rho(G)\,/\binom{n}{2} = \max_{B\in\mathcal{M}_F} g_\rho(B).$$

Proof. By Lemma 4.38,

$$\limsup_{\substack{v(G)\to\infty \\ F\not\subseteq G}} w_\rho(G)\,/\binom{n}{2} = g_\rho(B^*_{\rho,F}).$$

By Proposition 4.41, we have $B^*_{\rho,F} \in \mathcal{M}_F$. Furthermore, for all $B \in \mathcal{M}_F$, B is F-free, thus Lemma 4.38 implies $g_\rho(B) \leq g_\rho(B^*_{\rho,F})$. Hence

$$\max_{B\in\mathcal{M}_F} g_\rho(B) = g_\rho(B^*_{\rho,F}),$$

and the theorem is proved.

□

5. A variational characterization of $\theta(F)$

In this section, we use the main result of Section 4 to give a variational characterization of $\theta(F)$ as the solution to a finite-dimensional optimization problem. We essentially show that we can find $\theta(F)$ by computing the maximum of a certain function over some finite set of mixed adjacency matrices. This is a useful method to find the precise value of $\theta(F)$ in the general case, where Theorem 1.8 does not provide a tight bound.

We first present an easy analytical lemma which essentially confirms that the maximal ρ-weighted edge density of a blowup is continuous in ρ; this is virtually identical to the argument in Proposition 2.2.

Lemma 5.1 (Continuity lemma)**.** *Let A be a mixed adjacency matrix. Then $g_\rho(A)$ is continuous in ρ for $\rho \in (1,\infty)$.*

Proof. Let $A = (U, D)$. Note that $\mathbf{y}^\intercal D\mathbf{y} \leq 1$ for any $\mathbf{y} \in \triangle^{r-1}$, hence for all $\epsilon > 0$ we have

$$g_\rho(A) \leq g_{\rho+\epsilon}(A) = \max_{\mathbf{y}\in\triangle^{r-1}} \mathbf{y}^\intercal(U + (\rho+\epsilon)D)\mathbf{y} \leq \max_{\mathbf{y}\in\triangle^{r-1}} (\mathbf{y}^\intercal(U+\rho D)\mathbf{y} + \epsilon) \leq g_\rho(A) + \epsilon,$$

and similarly $g_\rho(A) \geq g_{\rho-\epsilon}(A) \geq g_\rho(A) - \epsilon$. Continuity follows directly.

□

Remark 5.2. In fact $g_\rho(A)$ can be shown to be convex, but we will not need this result here.

Theorem 5.3. *Let F be a mixed graph with at least one directed edge such that $\theta(F) \in (1, \infty)$. For each mixed adjacency matrix $B = (U, D)$ of size r in $\mathcal{M}_F \setminus \{K\}$ we let*

$$f(B) = \min_{\mathbf{y} \in \triangle^{r-1}} \left(\frac{1 - \mathbf{y}^\intercal U \mathbf{y}}{\mathbf{y}^\intercal D \mathbf{y}} \right),$$

then $\theta(F) = \min_{B \in \mathcal{M}_F \setminus \{K\}} f(B)$. (If $\mathbf{y}^\intercal D \mathbf{y} = 0$, we consider $\frac{1-\mathbf{y}^\intercal U \mathbf{y}}{\mathbf{y}^\intercal D \mathbf{y}} = \infty$.)

Proof. We break this into two claims.

Claim 5.4. For any $B = (U, D)$ of size r in $\mathcal{M}_F \setminus \{K\}$ and $\mathbf{y} \in \triangle^{r-1}$, we have

$$\theta(F) \leq \frac{1 - \mathbf{y}^\intercal U \mathbf{y}}{\mathbf{y}^\intercal D \mathbf{y}}.$$

Proof. Suppose otherwise. Then there exists mixed adjacency matrix B and vector $\mathbf{y} \in \triangle^{r-1}$ such that $\mathbf{y}^\intercal U \mathbf{y} + \theta(F) \mathbf{y}^\intercal D \mathbf{y} > 1$. But this means $g_{\theta(F)}(B) > 1$, which by Theorem 4.42 contradicts the definition of $\theta(F)$.

□

Claim 5.5. There exists $B = (U, D)$ of size r in $\mathcal{M}_F \setminus \{K\}$ and $\mathbf{y} \in \triangle^{r-1}$ such that

$$\theta(F) = \frac{1 - \mathbf{y}^\intercal U \mathbf{y}}{\mathbf{y}^\intercal D \mathbf{y}}.$$

Proof. Applying Theorem 4.42 to $\rho_k = \theta(F) + \frac{1}{k}$ for each positive integer k, we obtain for each k a particular extremal mixed adjacency matrix $B_k \in \mathcal{M}_F$, such that

$$\limsup_{\substack{v(G) \to \infty \\ F \not\subseteq G}} \alpha(G) + \rho_k \beta(G) = g_{\rho_k}(B_k). \tag{5.1}$$

Since $\rho_k > \theta(F)$, we have $\limsup_{\substack{v(G) \to \infty \\ F \not\subseteq G}} \alpha(G) + \rho_k \beta(G) > 1$ by the definition of $\theta(F)$. Therefore $g_{\rho_k}(B_k) > 1$, hence $B_k \neq K$. Since $\mathcal{M}_F$ is finite, there must exist some matrix B^* and an infinite sequence $k_1 < k_2 < \cdots$ such that $B^* = B_{k_1} = B_{k_2} = \cdots$. It is clear that $B^* \in \mathcal{M}_F \setminus \{K\}$.

We may now apply Lemma 5.1 on B^* and the sequence ρ_k to conclude that

$$g_{\theta(F)}(B^*) = \lim_{k \to \infty} g_{\rho_k}(B^*) \geq 1.$$

But by Theorem 4.42, $g_{\theta(F)}(B^*) \leq 1$, so $g_{\theta(F)}(B^*) = 1$ and the conclusion follows. □

Combining the two claims, we are done. □

6. The algebraicity of $\theta(F)$

In this section, we apply Theorem 5.3 to draw additional conclusions about the values $\theta(F)$ can take on. We will first prove Theorem 1.9 by explicit construction. Then, we will present a similar, more general argument that shows $\theta(F)$ must be algebraic for all F.

6.1. **A mixed graph F with irrational $\theta(F)$.**

Proof of Theorem 1.9. Let F be the mixed graph in Figure 11; we show that $\theta(F) = 1 + \frac{1}{\sqrt{2}}$.

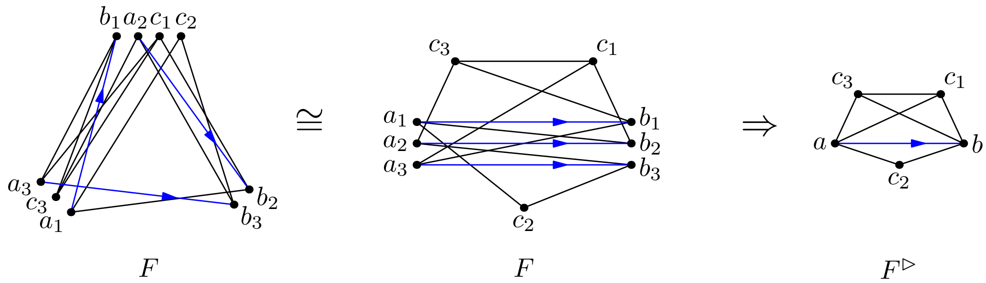


Figure 11

From inspection of Figure 11 we see $\chi\left(\widetilde{F}\right) = 3$ and $\chi\left(\widetilde{F^{\triangleright}}\right) = 4$. Define the three-vertex mixed graphs $G_1 = \{x\check{y}, x\check{z}, yz\}$, $G_2 = \{x\check{y}, x\check{z}, y\check{z}\}$, $G_3 = \{x\check{y}, y\check{z}, z\check{x}\}$, and $G_4 = \{z\check{x}, x\check{y}, yz\}$; see Figure 12. It is clear from Figure 13 that $F \subseteq G_1[3], F \subseteq G_2[3]$ and from Figure 14 that $F \subseteq G_3[3]$.

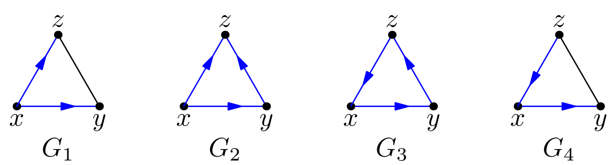


Figure 12

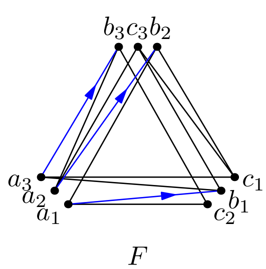


Figure 13.
$F \subseteq G_1[3]$

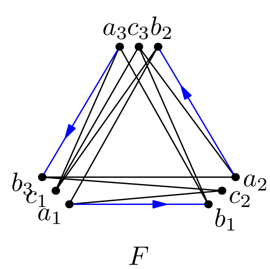


Figure 14.
$F \subseteq G_3[3]$

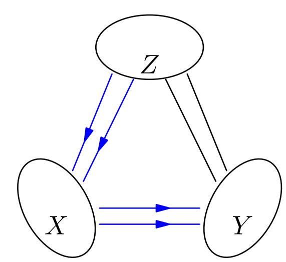


Figure 15.
$F \not\subseteq G_4[t]$

On the other hand we claim $F \not\subseteq G_4[t]$ for all positive integers t. For the sake of contradiction, assume otherwise; let X, Y, Z represents the three vertex sets obtained from blowing up the vertices x, y, z, respectively (see Figure 15). Because a_1 is the tail vertex in the edge $a_1\check{b}_1$, it must be in either X or Z.

If $a_1 \in X$, then the edge $a_1\check{b}_1$ forces $b_1 \in Y$. The edge $a_1 b_2$ forces $b_2 \notin X$; since b_2 is a head vertex, $b_2 \in Y$. Then the edge $a_2\check{b}_2$ forces $a_2 \in X$; the edge $a_2 b_3$ forces $b_3 \notin X$, so

$b_3 \in Y$ since b_3 is a head vertex. This in turn forces $a_3 \in X$. Finally, because c_1b_2 and c_1a_3 are edges, $c_1 \in Z$; similarly $c_3 \in Z$. But c_1c_3 is an edge, contradiction.

Alternatively, if $a_1 \in Z$, then $b_1 \in X$. The edge b_1a_3 forces $a_3 \notin X$; since a_3 is a tail vertex, we have $a_3 \in Z$ and therefore $b_3 \in X$. Next, b_3a_2 is an edge, so $a_2 \notin X$. Since a_2 is a tail vertex, $a_2 \in Z$, so $b_2 \in X$. Then, as in case 1, we must have $c_1, c_3 \in Y$, contradiction.

We may now apply Theorem 5.3 to find the exact value of $\theta(F)$. The matrices we need to consider are the F-free matrices with at most $\chi\left(\widetilde{F^{\triangleright}}\right) - 1 = 3$ vertices:

U	D	$\mathbf{x}^*$	$\min\limits_{\mathbf{x}\in\triangle^{r-1}} \dfrac{1-\mathbf{x}^\intercal U\mathbf{x}}{\mathbf{x}^\intercal D\mathbf{x}}$
$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&2\\0&0\end{bmatrix}$	$\langle\frac{1}{2},\frac{1}{2}\rangle$	2
$\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&2&0\\0&0&2\\0&0&0\end{bmatrix}$	$\left\langle 1-\frac{1}{\sqrt{2}}, \sqrt{2}-1, 1-\frac{1}{\sqrt{2}}\right\rangle$	$1+\frac{1}{\sqrt{2}}$
$\begin{bmatrix}0&0&1\\0&0&1\\1&1&0\end{bmatrix}$	$\begin{bmatrix}0&2&0\\0&0&0\\0&0&0\end{bmatrix}$	$\langle\frac{1}{2},\frac{1}{2},0\rangle$	2

Hence
$$\theta(F) = \min_{A\in\mathcal{M}_F\setminus\{K\}} \min_{\mathbf{y}\in\triangle^{r-1}} \left(\frac{1-\mathbf{y}^\intercal U\mathbf{y}}{\mathbf{y}^\intercal D\mathbf{y}}\right) = 1 + \frac{1}{\sqrt{2}}.$$
□

6.2. $\theta(F)$ **is always algebraic.**

Proof of Theorem 1.10. Let B^* be the mixed adjacency matrix in $\mathcal{M}_F \setminus \{K\}$ described in Theorem 5.3, so $g_{\theta(F)}(B^*) = 1$. Let $B^* \xrightarrow{\text{sub}} B$ with respect to $\rho = \theta(F)$, so $g_\rho(B) = 1$ also. Let r be the size of B. By Lemma 4.16 that there exists a unique $\mathbf{y} \in \triangle^{r-1}$ such that

$$B_\rho^{\text{sym}}\mathbf{y} = \lambda\mathbf{1}, \tag{6.1}$$

and $\mathbf{y}$ has all positive coordinates. Let C be the $(r-1)\times r$ matrix

$$C = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

and observe that $\|\mathbf{y}\|_1 = 1$ if and only if $\mathbf{1}\cdot\mathbf{y} = 1$, and (6.1) holds if and only if $C\left(B_\rho^{\text{sym}}\right)\mathbf{y} = 0$. Hence, defining the $r \times r$ matrix

$$A = \begin{bmatrix} 1 \; 1 \; \cdots \; 1 \\ C(B_\rho^{\text{sym}}) \end{bmatrix}$$

we deduce $A\mathbf{y} = \langle 1, 0, \ldots, 0\rangle$ has a unique solution $\mathbf{y} \in \triangle^{r-1}$. This implies A is invertible, hence $\mathbf{y} = A^{-1}\langle 1, 0, \ldots, 0\rangle$. Simple calculation shows that all elements of A are in the set $\{0, \pm 1, \pm\rho, \pm(\rho - 1)\}$. Hence all elements of A^{-1} can be expressed as $\frac{P(\rho)}{Q(\rho)}$ where P, Q are polynomials in $\mathbb{Z}[\rho]$, which implies the same for $\mathbf{y}$, the first column of A^{-1}. Finally, by Theorem 5.3 we have

$$\theta(F) = \rho = \frac{1 - \mathbf{y}^\intercal U \mathbf{y}}{\mathbf{y}^\intercal D \mathbf{y}},$$

which means $\theta(F)$ is algebraic as claimed. □

Appendix A. Proof of Proposition 3.1

Proof of Proposition 3.1. This is equivalent to showing $e_{\text{u}}(G) + \frac{\binom{n}{2}}{t(n,r)} e_{\text{d}}(G) \leq \binom{n}{2}$. Take a mixed graph G that maximizes $e_{\text{u}}(G) + \frac{\binom{n}{2}}{t(n,r)} e_{\text{d}}(G)$ over all n-vertex $\overrightarrow{K_{r+1}}$-free graphs.

We first use Zykov symmetrization [1] to show that for any unconnected a, b, we may assume a and b have the same neighborhood, i.e. for all vertices v either va, vb are both nonedges or are identically oriented with respect to v.

Claim A.1 (Zykov Symmetrization). Suppose a, b are unconnected by any edge in G. Then there exists G' on the same set of vertices as G such that G' is $\overrightarrow{K_{r+1}}$-free, a, b have the same neighborhood in G', and $e_{\text{u}}(G') + \frac{\binom{n}{2}}{t(n,r)} e_{\text{d}}(G') = e_{\text{u}}(G) + \frac{\binom{n}{2}}{t(n,r)} e_{\text{d}}(G)$.

Proof. Without loss of generality, assume $\deg_{\text{u}}(a) + \frac{\binom{n}{2}}{t(n,r)} \deg_{\text{d}}(a) \leq \deg_{\text{u}}(b) + \frac{\binom{n}{2}}{t(n,r)} \deg_{\text{d}}(b)$. Delete a and replace it with a perfect copy b' of b. This does not decrease the value of $e_{\text{u}}(G) + \frac{\binom{n}{2}}{t(n,r)} e_{\text{d}}(G)$, and it does not introduce a copy of $\overrightarrow{K_{r+1}}$, since no copy was originally in G, and no copy can contain both vertices b and b' because they are not connected. □

Consequently, assume without loss of generality that Claim A.1 holds for all $a, b \in V(G)$. Let A be a maximal independent set of G; every $v \in V(G) \setminus A$ is connected to at least one vertex in A, and thus by Claim A.1 is connected to all vertices in A, with va identically oriented for all $a \in A$. Partition $V(G) = A \sqcup B \sqcup C$, where B is the set of vertices connected to A by undirected edges and C is the set of vertices connected to A by directed edges (see Figure 16). (see Figure 16)

We now induct on r. For the base case $r = 2$ note that B and C cannot both be nonempty, else a $\overrightarrow{K_3}$ is formed by choosing one vertex from each. If B is empty, then there are no undirected edges in C (if $c_1\check{c}_2$ is such an edge then $\{a, c_1, c_2\}$ for $a \in A$ forms a $\overrightarrow{K_3}$),

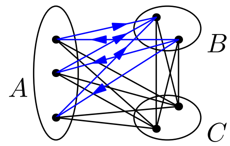


Figure 16

so G has no directed edges; the conclusion reduces to $e_{\mathrm{u}}(G) \le \binom{n}{2}$ which is apparent. If C is empty, note that there are no edges in B (if b_1b_2 is an edge then $\{a, b_1, b_2\}$ for $a \in A$ forms a $\overrightarrow{K_3}$) and therefore G is bipartite. Thus by Turán's theorem $e(G) \le t(n, 2)$ and the result follows.

For the inductive step, assume that this proposition is true for $\overrightarrow{K_r}$-free graphs; we will show it is true for $\overrightarrow{K_{r+1}}$-free graphs. Note that any vertices $u \in B$ and $v \in C$ are connected by some edge, since u and v do not have the same neighborhood. Now we casework on B:

If B is empty, note that C is $\overrightarrow{K_r}$-free (if $\overrightarrow{K_r} \in G[C]$ then adding $a \in A$ forms a $\overrightarrow{K_{r+1}}$), so

$$e_{\mathrm{u}}(G[C]) + \frac{\binom{n}{2}}{t(n,r)} e_{\mathrm{d}}(G[C]) \le e_{\mathrm{u}}(G[C]) + \frac{\binom{n}{2}}{t(n,r-1)} e_{\mathrm{d}}(G[C]) \le \binom{|C|}{2},$$

the second inequality following from the inductive hypothesis. Thus

$$e_{\mathrm{u}}(G) + \frac{\binom{n}{2}}{t(n,r)} e_{\mathrm{d}}(G) \le |A| \cdot |C| + \binom{|C|}{2} < \binom{n}{2}.$$

If B is nonempty, then we claim that $K_{r+1} \not\subseteq G$. Else, there exists a set S of $r + 1$ pairwise-connected vertices. If S does not contain a vertex in A, then add one arbitrarily; do the same for B. Note that the new set S' has size at least $r + 1$ and vertices pairwise connected; furthermore for $u \in S' \cap A$ and $v \in S' \cap B$, the edge $uv \in E(G[S'])$ is directed. This implies $\overrightarrow{K_{r+1}} \subseteq G[S']$, contradiction. This claim being established, we easily have by Turán's theorem

$$e_{\mathrm{u}}(G) + \frac{\binom{n}{2}}{t(n,r)} e_{\mathrm{d}}(G) \le \frac{\binom{n}{2}}{t(n,r)} \left(e_{\mathrm{u}}(G) + e_{\mathrm{d}}(G)\right) \le \binom{n}{2}.$$

This completes the induction and the proof. □

Appendix B. Verification of Example 3.12

Let G the mixed graph in Figure 17.

We claim that $F \not\subseteq G[t]$ for all positive integers t. For the sake of contradiction assume $F \subseteq G[t]$ for some t. We denote the 4 parts of vertices of $G[t]$ by X, Y, Z, W. Because c_1, c_2, c_3 are pairwise connected, they must be in different parts of $G[t]$. Because of the symmetry in G, without loss of generality we only need to check two cases:

(i) c_1, c_2, c_3 are in X, Y, Z, respectively, forming a transitive tournament, see Figure 18;
(ii) c_1, c_2, c_3 are in W, Y, Z, respectively, forming a directed cycle, see Figure 19.

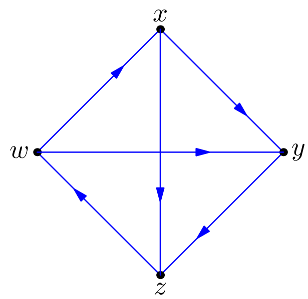


Figure 17

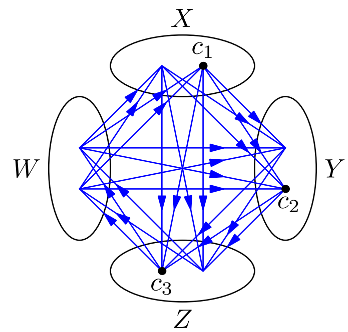


Figure 18

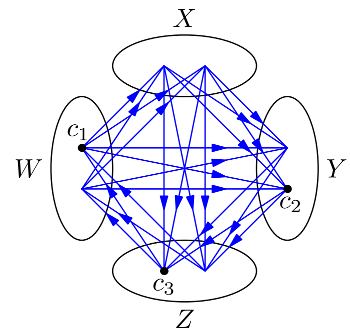


Figure 19

Case 1. $c_1 \in X, c_2 \in Y, c_3 \in Z$. Consider vertices b_1 and a_2. Because b_1 is connected to both c_2 and c_3, we know that $b_1 \notin Y$ and $b_1 \notin Z$. Similarly, $a_2 \notin X$ and $a_2 \notin Z$. If $b_1 \in W$, then $a_2 \notin W$ which forces $a_2 \in Y$, but that is impossible because the edge $a_2\check{b}_1$ has the wrong orientation. Therefore $b_1 \in X$. Similarly, consider a_2 and b_3, we get $a_2 \in Y$. But $b_1 \in X$ and $a_2 \in Y$ result in the wrong edge orientation for $a_2\check{b}_1$, contradiction.

Case 2. $c_1 \in W, c_2 \in Y, c_3 \in Z$. Using the same reasoning as in case 1, considering a_2 and b_1 we get $a_2 \in Y$. Then considering b_1 and a_3 we get $b_1 \in W$, again resulting in the wrong edge orientation for $a_2\check{b}_1$, contradiction.

Therefore $F \nsubseteq G[t]$ for all positive integers t. Note that $\alpha(G[t]) = 0$, and $\beta(G[t]) = 6t^2/\binom{4t}{2}$ tends to $\frac{3}{4}$ as t grows large, so $\theta(F) \le \frac{4}{3}$. Also by Theorem 1.8, $\theta(F) \ge 1 + \frac{1}{\chi(\tilde{F})} = \frac{4}{3}$. Hence $\theta(F) = \frac{4}{3}$.

Appendix C. Claims in Lemma 4.16

Proof of Claim 4.17. We apply the method of Lagrange multipliers. Compute

$$\nabla \mathbf{y}^\intercal A_\rho \mathbf{y} = \left(A_\rho + A_\rho^\intercal\right)\mathbf{y} = 2\left(A_\rho^{\mathrm{sym}}\right)\mathbf{y} \quad , \quad \nabla\|\mathbf{y}\|_1 = \mathbf{1}.$$

Hence a maximizing vector $\mathbf{y}^* \in \triangle^{r-1}$ of $\mathbf{y}^\intercal A_\rho \mathbf{y}$ satisfies

$$0 = \left(\nabla \mathbf{y}^\intercal A_\rho \mathbf{y} - \lambda \nabla \|\mathbf{y}\|_1\right)\Big|_{\mathbf{y}=\mathbf{y}^*} = 2\left(A_\rho^{\mathrm{sym}}\right)\mathbf{y}^* - \lambda \mathbf{1}$$

for some constant λ. Note that by definition $g_\rho(A) = (\mathbf{y}^*)^\intercal A_\rho \mathbf{y}^*$, hence

$$\lambda = (\mathbf{y}^*)^\intercal(\lambda\mathbf{1}) = 2(\mathbf{y}^*)^\intercal\left(A_\rho^{\mathrm{sym}}\right)\mathbf{y}^* = 2(\mathbf{y}^*)^\intercal A_\rho \mathbf{y}^* = 2g(A),$$

which shows $\mathbf{y}^*$ is a solution of (4.1). □

Proof of Claim 4.18. If $\mathbf{y}^*$ is a solution to (4.1) then

$$(\mathbf{y}^*)^\intercal A_\rho \mathbf{y}^* = (\mathbf{y}^*)^\intercal (A_\rho^{\mathrm{sym}})\mathbf{y}^* = (\mathbf{y}^*)^\intercal \cdot g_\rho(A)\mathbf{1} = g_\rho(A),$$

the last equality being true since $\|\mathbf{y}^*\|_1 = 1$.

We first show that no solution can have coordinates equal to 0. Suppose for the sake of contradiction that (4.1) has a solution $\mathbf{y}_0$ with some coordinate equal to 0. Let $\mathbf{y}'$ be the $(r-1)$-dimensional vector with that coordinate removed from $\mathbf{y}_0$, and let A' be the principal submatrix of A with the row and column of its undirected and directed parts corresponding to that coordinate removed. Then $(\mathbf{y}')^\intercal A'_\rho \mathbf{y}' = \mathbf{y}_0^\intercal A_\rho \mathbf{y}_0 = g_\rho(A)$, contradicting the fact that A is condensed.

Now we show that there cannot be more than one solution. Suppose for the sake of contradiction that there there are two distinct vectors with nonnegative coordinates which both satisfy (4.1), which means any affine combination of them does as well. A suitable combination produces a vector satisfying (4.1) with all coordinates nonnegative and at least one coordinate 0, reducing to the first case. □

Proof of Claim 4.19. By Proposition 4.6 and Proposition 4.10,

$$\frac{1}{2}\left(\mathbf{x}_{\rho,A}^{(n)}\right)^\intercal A_\rho \left(\mathbf{x}_{\rho,A}^{(n)}\right) = w_\rho\left(A\left[\!\left[\mathbf{x}_{\rho,A}^{(n)}\right]\!\right]\right) + O(n) = \frac{n^2}{2} g_\rho(A) + o(n^2). \tag{C.1}$$

Let $n_1 < n_2 < \cdots$ be a sequence of integers such that $\lim_{k\to\infty} \frac{1}{n_k}\mathbf{x}_{\rho,A}^{(n_k)} = \hat{\mathbf{y}}$. Then (C.1) implies that $\hat{\mathbf{y}}^\intercal A_\rho \hat{\mathbf{y}} = g_\rho(A)$. Now for all integer k, write $\mathbf{x}_{\rho,A}^{(n_k)} = \left(x_1^{(n_k)}, \ldots, x_r^{(n_k)}\right)$, and let v_1, v_2 be two vertices in $A\left[\!\left[\mathbf{x}_{\rho,A}^{(n_k)}\right]\!\right]$. By Lemma 4.15, all weighted degrees of the vertices are equal up to a difference of at most ρ. Using notation from Definition 4.3, assume without loss of generality that $v_1 \in C_1$ and $v_2 \in C_2$. Let $A = (U, D)$, then $\deg_\rho v_1 = \deg_\rho v_2 + O(1)$ can be written as

$$\sum_{j=1}^{r} \left(A_\rho^{\mathrm{sym}}\right)_{1j} x_j^{(n_k)} = \sum_{j=1}^{r} \left(A_\rho^{\mathrm{sym}}\right)_{2j} x_j^{(n_k)} + O(1)$$

Dividing both sides by n_k and taking the limit as $k \to \infty$ yields

$$\sum_{j=1}^{r} \left(A_\rho^{\mathrm{sym}}\right)_{1j} y_j = \sum_{j=1}^{r} \left(A_\rho^{\mathrm{sym}}\right)_{2j} y_j,$$

which means the first two coordinates of the vector $\left(A_\rho^{\mathrm{sym}}\right)\hat{\mathbf{y}}$ are equal. Repeating this reasoning all coordinates in the vector are equal, i.e. $\left(A_\rho^{\mathrm{sym}}\right)\hat{\mathbf{y}} = (c, c, \ldots, c)$ for some constant c. Since $g_\rho(A) = \hat{\mathbf{y}}^\intercal A_\rho \hat{\mathbf{y}} = \hat{\mathbf{y}}^\intercal (A_\rho)^\intercal \hat{\mathbf{y}}$, we have

$$g_\rho(A) = \hat{\mathbf{y}}^\intercal (A_\rho^{\mathrm{sym}})\hat{\mathbf{y}} = \hat{\mathbf{y}}^\intercal (c, c, \ldots, c) = c \cdot \|\hat{\mathbf{y}}\|_1 = c,$$

which means indeed $(A_\rho^{\mathrm{sym}})\hat{\mathbf{y}} = g_\rho(A) \cdot \mathbf{1}$, so (4.1) holds for $\hat{\mathbf{y}}$. □

Appendix D. Proof of Lemma 4.25

Proof of Lemma 4.25. We will assume N to be a fixed large integer whose value will be chosen later in Claim D.3. For any mixed graph G on n vertices that satisfies the lemma conditions, divide its vertices into two parts: V_1, the vertices of $\mathfrak{G}_{\rho,A}^{(N)}$, and V_2, the remaining

$n-N$ vertices. Further divide $V_1 = C_1 \sqcup \cdots \sqcup C_r$ (where r is the size of A) according to Definition 4.3. The partition of the vertices of G is illustrated in Figure 20.

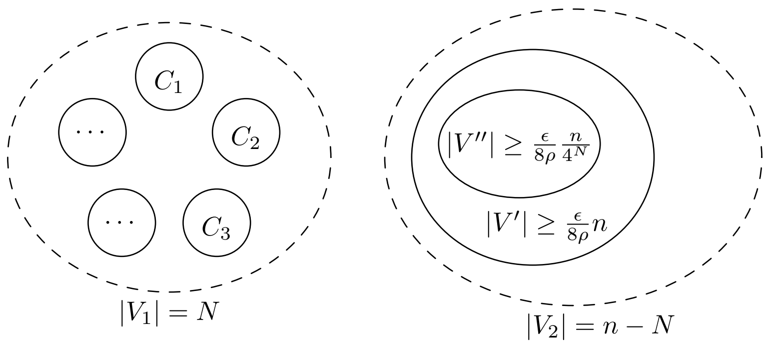


Figure 20

For vertex $v \in V(G)$ and vertex set $S \subseteq V(G)$, we use $\deg_\rho^{(S)}(v)$ to denote the weighted degree of v with respect to S, by computing $\deg_\rho(v)$ after removing all vertices not in $S \cup \{v\}$. Also define the weighted edge count between V_1 and V_2 be the sum of weighted edges between vertices of the components, $\sum_{v\in V_2} \deg_\rho^{(V_1)} v = \sum_{u \in V_1} \deg_\rho^{(V_2)} u$.

Claim D.1. Let $V' \subseteq V_2$ be the set of all vertices $v \in V_2$ such that $\deg_\rho^{(V_1)}(v) \geq (g_\rho(A) + \frac{\epsilon}{2})N$. Then $|V'| \geq \frac{\epsilon}{8\rho}n$ for sufficiently large $n = n(\epsilon, \rho, N)$, and any mixed graph G on n vertices satisfying the two conditions of the lemma.

Proof. By condition (i), the weighted edge count between V_1 and V_2 is

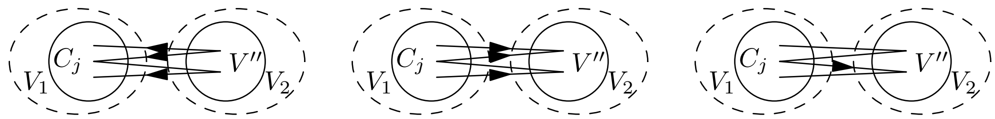
$$\sum_{v\in V_2} \deg_\rho^{(V_1)} v = \sum_{u\in V_1} \deg_\rho^{(V_2)} u \geq (g_\rho(A)+\epsilon)(n-N)N - \rho N.$$

When $n = n(\epsilon, \rho, N)$ is sufficiently large, $(g_\rho(A)+\epsilon)(n-N)N - \rho N \geq (g_\rho(A)+\frac{3\epsilon}{4})(n-N)N$. Also note $\deg_\rho^{(V_1)}(v) \leq \rho N$ for all $v \in V_2$. Thus

$$\begin{aligned}\left(g_\rho(A) + \frac{3\epsilon}{4}\right)(n-N)N &\leq \sum_{v\in V_2} \deg_\rho^{(V_1)} v \\ &= \sum_{v\in V_2\setminus V'} \deg_\rho^{(V_1)} v + \sum_{v \in V'} \deg_\rho^{(V_1)} v \\ &\leq \left(g_\rho(A) + \frac{\epsilon}{2}\right) N(n-N) + \rho N|V'|,\end{aligned}$$

which simplifies to $|V'| \geq \frac{\epsilon}{4\rho}(n-N) \geq \frac{\epsilon}{8\rho}n$. □

Note there are only 4^N ways for a vertex in V_2 to be connected to vertices in V_1. Hence there exists $V'' \subseteq V'$ of at least $\frac{\epsilon}{8\rho}\frac{n}{4^N}$ vertices, for which all vertices $v \in V''$ are connected identically to vertices in V_1, and $\deg_\rho^{(V_1)} v \geq \left(g_\rho(A) + \frac{\epsilon}{2}\right) N$.



(i) (ii) (iii)

Figure 21

Let $A = (U, D)$. We define matrix $B = (U', D')$ of size $r + 1$ as follows: the first r rows and r columns of U' and D' are identical to those of U and D respectively, $U'_{(r+1)(r+1)} = D'_{(r+1)(r+1)} = 0$, and for each $j \in [r]$ (see Figure 21 for illustration):

(i) if there are at least $\frac{\epsilon}{8r\rho}N$ vertices in C_j joined to each vertex in V'' by a directed edge, with head vertex in C_j, set $D'_{(r+1)j} = 2$ and $U'_{(r+1)j} = U'_{j(r+1)} = D'_{j(r+1)} = 0$;

(ii) else if there are at least $\frac{\epsilon}{8r\rho}N$ vertices in C_j joined to each vertex in V'' by a directed edge, with head vertex in V'', set $D'_{j(r+1)} = 2$ and $U'_{(r+1)j} = U'_{j(r+1)} = D'_{(r+1)j} = 0$;

(iii) else if there are at least $\frac{\epsilon}{8r\rho}N$ vertices in C_j joined to each vertex of V'' (by either directed or undirected edges), set $U'_{(r+1)j} = U'_{j(r+1)} = 1$ and $D'_{(r+1)j} = D'_{j(r+1)} = 0$;

(iv) else, set $U'_{(r+1)j} = U'_{j(r+1)} = D'_{(r+1)j} = D'_{j(r+1)} = 0$.

Claim D.2. For vertex $v \in V''$ and $j \in [r]$, the weighted degree of v with respect to C_j is

$$\deg_\rho^{(C_j)} v \leq \left(B_\rho^{\text{sym}}\right)_{(r+1)j} |C_j| + \frac{\epsilon N}{4r}. \tag{D.1}$$

Proof. We examine all four cases in the construction of B above. Since $\rho > 1$, $\deg_\rho^{(C_j)} v \leq \rho|C_j|$, so in cases (i) and (ii), $\left(B_\rho^{\text{sym}}\right)_{(r+1)j} = \rho$, thus (D.1) is true. In case (iii), $\left(B_\rho^{\text{sym}}\right)_{(r+1)j} = 1$, and the number of directed edges is less than $2\frac{\epsilon}{8r\rho}N$, therefore

$$\deg_\rho^{(C_j)}(v) < |C_j| + (\rho - 1)\left(2\frac{\epsilon}{8r\rho}N\right) < |C_j| + \frac{\epsilon N}{4r}.$$

Finally in case (iv), the number of edges (directed or undirected) is less than $\frac{\epsilon}{8r\rho}N$, hence $\deg_\rho^{(C_j)} v < \rho\frac{\epsilon}{8r\rho}N < \frac{\epsilon N}{4r}$. □

Claim D.3. When $N = N(A, \rho, \epsilon)$ is sufficiently large, $A \xrightarrow{\text{aug}} B$.

Proof. By Claim D.2, for each vertex $v \in V''$, the weighted degree of v with respect to V_1 is

$$\deg_\rho^{(V_1)} v \leq \sum_{j=1}^{r} \left(B_\rho^{\text{sym}}\right)_{(r+1)j} |C_j| + \frac{\epsilon N}{4}.$$

Combining this with $\deg_\rho^{(V_1)} v \geq (g_\rho(A) + \frac{\epsilon}{2})N$, we get

$$\left(g_\rho(A) + \frac{\epsilon}{2}\right) N \leq \deg_\rho^{(V_1)} v \leq \sum_{j=1}^{r} \left(B_\rho^{\text{sym}}\right)_{(r+1)j} |C_j| + \frac{\epsilon N}{4},$$

Dividing by N,

$$g_\rho(A) + \frac{\epsilon}{4} \le \sum_{j=1}^{r} \left(B_\rho^{\mathrm{sym}}\right)_{(r+1)j} \frac{|C_j|}{N}.$$

Let $\mathbf{y}^*_{\rho,A} = \langle y_1, \dots, y_r \rangle$ be the optimal vector of A. Recall that C_j are the parts of $\mathfrak{G}^{(N)}_{\rho,A}$, so $\lim_{N\to\infty} \frac{|C_j|}{N} = y_j$ by Lemma 4.16. So when $N = N(A, \rho, \epsilon)$ is sufficiently large, we have

$$g_\rho(A) < \sum_{j=1}^{r} \left(B_\rho^{\mathrm{sym}}\right)_{(r+1)j} y_j,$$

which means $A \xrightarrow{\text{aug}} B$ by Definition 4.23. $\square$

Finally, we'll show that $\mathfrak{G}^{(m)}_{\rho,B} \subseteq G$. Because A is condensed, by Corollary 4.21, if N is sufficiently large, then $A\,[\![m\mathbf{1}]\!] \subseteq \mathfrak{G}^{(N)}_{\rho,A}$. Thus $A\,[\![m\mathbf{1}]\!] \subseteq \mathfrak{G}^{(N)}_{\rho,A} \subseteq G$ by condition (ii) of the lemma. By construction of B, when n is sufficiently large, $|V''| \ge m$ and all edges of $B\,[\![m\mathbf{1}]\!]$ corresponding to the $(r+1)$th row and $(r+1)$th column of the undirected and directed parts of B are in G. Therefore $B\,[\![m\mathbf{1}]\!] \subseteq G$. Since $\mathfrak{G}^{(m)}_{\rho,B} \subseteq B\,[\![m\mathbf{1}]\!]$, we have $\mathfrak{G}^{(m)}_{\rho,B} \subseteq G$. $\square$

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