S. T. YAU HIGH SCHOOL SCIENCE AWARD

Research Report

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Title of Research Report

Statistics of Low-Lying Zeros in Families of Cuspidal Newforms

Date

August 20, 2023

STATISTICS OF LOW-LYING ZEROS IN FAMILIES OF CUSPIDAL NEWFORMS

XUEYIMING TANG

ABSTRACT. Assuming the generalized Riemann Hypothesis, the non-trivial zeros of L-functions lie on the critical line with the real part equals $1/2$. We seek answers to the following natural question: where on the critical line do we see the first zero? Previous studies have focused on the first zero of L-functions in terms of how high it could be. Bober et al. proved that the first zero $1/2+i\gamma$ for a general L-function has $-\gamma_0 < \gamma < \gamma_0$, where $\gamma_0 \approx 22.27$. In this paper, we focus on finding the lower bound of the lowest first zero in families of cuspidal newforms of prime level tending to infinity. We obtain explicit bound through n-level densities derived from Random matrix theory and results towards the Katz-Sarnak density conjecture. We prove that as the level tends to infinity there is at least one form with first zero very close to the central point (roughly one hundred times closer than the bound from Bober et al.). We also obtain the first-ever bounds on the percentage of forms in the family with a fixed number of zeros within a small distance near the central point.

Keywords: L-functions, low-lying zeros, zero gaps, Random matrix theory.

Acknowledgement

I would like to express my deepest gratitude to Professor Steven Miller, whose invaluable guidance and insightful feedback have been instrumental in the completion of this research. I have learned tremendously throughout the process. I would also like to thank my supportive parents and the guidance from my teachers over the years.

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The participating team declares that the paper submitted is comprised of original research and results obtained under the guidance of the instructor. To the team's best knowledge, the paper does not contain research results, published or not, from a person who is not a team member, except for the content listed in the references and the acknowledgment. If there is any misinformation, we are willing to take all the related responsibilities.

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Date: August 19, 2023

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1. Introduction

1.1. **Background.** Prime numbers have long been lying at the heart of number theory and mathematics. From the RSA encryption system in cryptography to the hypothesized cicadas' hibernation behavior, they play vital roles in real-world applications. In particular, number theorists are interested in the distribution of primes as they encode deep mathematical information. One way to investigate the distribution of primes is through the Riemann zeta-function, which allows us to pass from integers (whose distribution we understand perfectly) to primes. We quickly review the zeta function and its properties and generalizations below. It is defined, for $\Re(s) > 1$, by

$$\zeta(s) \ := \ \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

Utilizing the fundamental theorem of arithmetic and the geometric series formula, a conventional argument demonstrates that we can represent $\zeta(s)$ as a product over primes (Euler formula):

$$\zeta(s) \ = \ \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}. \tag{1.2}$$

Thus the distribution of primes is closely connected to the zeros of the zeta-function; we can see this immediately as we have a sum over integers equal to a product over primes, and we can make this interplay more explicit by using techniques from complex analysis. Specifically, we do a contour integral of the logarithmic derivative of $\zeta(s)$, a standard technique in the subject; this highlights the importance of the Euler product: while the logarithmic derivative of the summation form is challenging to handle, the logarithm of a product simplifies to a sum of logarithms. This computation leads to the Prime Number Theorem (PNT), which states that $x/\log x$ is an excellent approximation of $\pi(x)$, the prime counting function. While we know $\pi(x) - x/\log x$ tends to zero, a natural question is how quickly? The answer involves the distribution of zeros of $\zeta(s)$ in the entire complex plane, and instead of using $x/\log x$ a better approximation is $\mathrm{Li}(x) := \int_2^x dt/\log t$.

Thus, to count primes, our first step is to find the analytic continuation of the zeta-function for the entire complex plane. This is accomplished through the functional equation

$$\zeta(s) \ = \ 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s)\zeta(1-s), \tag{1.3}$$

and we see there is a pole with residue 1 at $s = 1$, and $\zeta(s) = \zeta(1-s)$. Furthermore, this analytic continuation of ζ has zeros at the negative even integers, and does not have any other zeros outside $0 \leq \gamma \leq 1$. This interval is called the critical strip, and all the zeros outside the critical strip are known as trivial zeros. In 1859, Riemann famously proposed what is now called the Riemann Hypothesis (RH) in his honor, conjecturing that all non-trivial zeros of the zeta function are concentrated on a critical line where the real part equals $1/2$. If true, this implies that $|\pi(x) - \mathrm{Li}(x)|$ is on the order of $x^{1/2}\log x$; if instead the largest zero had real part θ then the difference would be approximately of size x^θ. This result motivates much of modern number theory: the distribution of zeros of functions can encode important arithmetic information.

Building on the success from studying the Riemann zeta function, other functions that share similar qualities are also worth investigating. These objects are known as L-functions, and they have Dirichlet series

$$L(s) \ = \ \sum_{n=1}^{\infty} a_n n^{-s}, \tag{1.4}$$

where the a_n are related to some object from number theory. These too initially converge in some region and for appropriately chosen coefficients have Euler products and satisfy a functional equation; we often normalize the coefficients so that the functional equation again relates values at s to those at $1-s$ (unfortunately this is not always done; for example much of the elliptic curve literature involves s going to $2-s$, and thus one must be careful when reading results from different sources).

These L-functions provide a way to study arithmetic objects analytically. The zeros and poles of L-functions carry substantial information, and since they are analytic objects, we are able to use methods from complex analysis to understand related arithmetically interesting objects. For instance, the L-functions derived from Dirichlet characters play an important role in demonstrating that an infinite number of primes exist within an arithmetic progression. The Birch and Swinnerton-Dyer Conjecture, referenced in [BSD1, BSD2] and recognized as one of the illustrious seven Millennium Prize Problems set by the Clay Math Institute, conjectured that the geometric rank of an elliptic curve's Mordell-Weil group is equivalent to the analytic rank of the corresponding L-function. This is contingent upon the Generalized Riemann Hypothesis (GRH), which states that all non-trivial zeros in the critical strip possess a real part of 1/2. It's worth noting that we invariably assume the GRH for any L-function we come across. As the distribution of zeros of L-functions is of great importance, a natural question is subsequently raised when we seek to understand the distribution of zeros. We focus on the following: *assuming the GRH, how far up must we go on the critical line before we are assured that we will see the first zero?*

Several studies have been done on the first zero of L-functions. The first critical zero of the Riemann zeta-function is surprisingly large compared to other L-functions, around $\frac{1}{2} + 14.135i$. Stephen D. Miller provides an upper bound on the first zero of the form $\frac{1}{2} + i\gamma$ for all L-functions of real archimedian type to be $-\gamma_0 < \gamma < \gamma_0$, where $\gamma_0 \approx 14.13$. The result is further generalized in [B–] to hold for general L-functions, for $-\gamma_1 < \gamma < \gamma_1$ and $\gamma_1 \approx 22.661$.

Complementing the previous studies on the highest of the lowest zero of L-functions, the goal of this paper is to study families of L-functions and investigate where the lowest of the lowest zero is. We briefly describe our results, which we will give in full detail after introducing some background material. By looking at a family of related L-functions we obtain significantly lower bounds for the first zero from some form in the family than must be true for all forms in the family.

Specifically, we prove lower bounds on the first zero above the central point among all the forms the family of even cuspidal newforms of prime level tending to infinity, though similar calculations would work for the family of odd cuspidal newforms with a trivial modification (since odd forms have a zero at the central point by symmetry, one simply removes that known contribution from the computations). Moreover, we obtain the first ever results on the number of zeros near the central point (low-lying zeros) by obtaining an upper bound on the percentage of forms in the family that have a certain number of zero in a small interval near the central point. We bound the percentages of forms in these families that can have "many" zeros "close" to the central point (the smaller the window and the greater the number of zeros the less likely this is to happen; we quantify how unlikely).

Below we describe the families of L-functions we study, and then state our results.

1.2. **Modular Forms.** To understand cuspidal newforms, we first must introduce the notion of modular forms and briefly revisit their definition and a few of their key characteristics. Given a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a complex number z, we set γz to be the the fractional linear transformations $\frac{az+b}{cz+d}$ that have a rich collection of properties. What will be beneficial for us is the behavior of certain choices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z \;=\; z+1, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z \;=\; -1/z. \tag{1.5}$$

These two matrices generate the group $\mathrm{SL}_2(\mathbb{Z})$, the set of 2×2 integer matrices with determinant 1; below we look at functions that transform nicely under this group, as well as subgroups of this group.

Definition 1.1. *Let f be a function on the upper-half of the complex plane $\mathcal{H} = z \in \mathbb{C} : \Im(z) > 0$. A modular form of weight k is an meromorphic function $f : \mathcal{H} \to \mathbb{C}$ such that*

$$f(\gamma z) \;=\; (cz+d)^k f(z). \tag{1.6}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ *(the set of two by two integer matrices with determinant 1). Additionally, f is bounded as $\Im(z)$ goes to infinity.*

Notice that if we take $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we have $f(z+1) = f(z)$, which means that f is periodic with period 1. The cusps forms are modular forms where $f(z)$ approaches 0 when $\Im(z) \to \infty$.

We can generalize the above definition through additional restrictions on matrices under which our functions should transform nicely. A modular form of level N has the additional property that we only care about the transformation rule for $f(\gamma z)$ when the lower left entry of γ (what we called c above) is equal to zero modulo N.

Definition 1.2 (Cuspidal Newforms)**.** *Let $H_k^\star(N)$ be the set of holomorphic cusp forms of weight k that are newforms of level N. For every $f \in H_k^\star(N)$, we have a Fourier expansion*

$$f(z) \ = \ \sum_{n=1}^{\infty} a_f(n) e(nz). \tag{1.7}$$

We set $\lambda_f(n) = a_f(n) n^{-(k-1)/2}$. The L-function associated to f is

$$L(s,f) \ = \ \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}. \tag{1.8}$$

The completed L-function is

$$\Lambda(s,f) \ := \ \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s,f). \tag{1.9}$$

Each $\Lambda(s,f)$ satisfies a functional equation $\Lambda(s,f) = \epsilon_f \Lambda(1-s,f)$, where ϵ_f is either +1 or -1. We can thus separate $H_k^\star(N)$ into two subsets:

$$H_k^+(N) \ = \ f \in H_k^\star(N) : \epsilon_f = +1 \quad \text{and} \quad H_k^-(N) \ = \ f \in H_k^\star(N) : \epsilon_f = -1.$$

We call the first set the even forms (and these will be the ones we study below), while the second are the odd ones. We adjusted the coefficients of our form by removing the factor $n^{-(k-1)/2}$ so that the functional equation will relate s to $1-s$.

Since the work of Montgomery and Dyson, there is now an extensive literature that many properties of L-functions, from their values to the distribution of spacings between their zeros, can be well modeled by ensembles of matrices (for details see §1.3). The symmetry group corresponding to $H_k^\star(N)$ is the Orthogonal group, denoted as *(O)*. For the subset $H_k^+(N)$, its associated symmetry group is the Special Orthogonal group of even order, represented as *SO(even)*, and for $H_k^-(N)$, its associated symmetry group is the Special Orthogonal group of odd order, represented as *SO(odd)*.

1.3. **Random Matrix Theory.** Hugh Montgomery's work in 1972 unveiled a fascinating link between the distribution of zeros of the Riemann zeta-function and the distribution of eigenvalues of random Hermitian matrices. This was a significant breakthrough in number theory and mathematical physics, as it suggested that the seemingly unrelated fields of prime number distribution and quantum physics could have deep underlying connections. Specifically, during a meeting with physicist Freeman Dyson, they observed that in the regime where they could prove results, the pair correlation of zeros of the Riemann zeta-function obey the same statistics as the eigenvalues of large random Hermitian matrices of Gaussian Unitary Ensemble, also known as GUE (these are complex Hermitian matrices where the entries are chosen from Gaussin distributions)[1] This observation led Montgomery to formulate what is now known as the Montgomery's

[1]The distribution of the independent entries must be Gaussian in order for the probability of a matrix A and a unitary conjugate, $U^{\mathrm{H}} A U$, to have the same probability of being chosen; this corresponds to the physical property that it is the eigenvalues of a matrix that matter, and not the matrix elements, which are functions of the choice of basis.

GUE conjecture, which posits that the n-level correlation function of the zeta function's zeros and that of the GUE coincide (later proven by Rudnick and Sarnk in 1996 for suitably restricted test functions for not just $\zeta(s)$ but for all automorphic L-functions). As our goal is to study other statistics than the n-level correlations, we just remark that this was the first statistic studied, and knowing these correlations for all n yields the spacing distribution between adjacent zeros. Odlyzko calculated the zeros of the zeta-function for the pair correlation at the height of $10^{20\text{th}}$ zero and observed that the spacing of nearby zeros closely resembles that of the GUE (see [H, Con] and their references for more details on this background on Random Matrix Theory).

The reason we move to other statistics is that although the n-level correlations provide amazing results of the distribution of zero of one form, these are defined through limits of sums over the zeros on the critical line and are thus insensitive to finitely many zeros. In particular, they cannot say anything about the behavior near the central point, which as remarked above is often of great arithmetic interest. Katz and Sarnak [KS] then observed that the statistics of zeros of many L-functions also align with the eigenvalue of classical compact groups other than the GUE ensemble of matrices. In particular, they introduced new statistics, the n-level densities, to better study these symmetries. The n-level density provides a finer statistical analysis of the zeros, especially closer to the central point, which the pair correlations and n-level correlations were unable to examine previously.

Definition 1.3 (n-Level Density)**.** *The n-level density of an L-function $L(s,f)$ for a* test function $\Phi : \mathbb{R}^n \to \mathbb{R}$ *is defined as*

$$D_n(f;\Phi) \ := \sum_{\substack{j_1,\cdots,j_n \\ j_i \neq \pm j_k}} \Phi\left(\frac{\log c_f}{2\pi}\gamma_f^{(j_1)},\dots,\frac{\log c_f}{2\pi}\gamma_f^{(j_n)}\right), \tag{1.10}$$

with c_f being the analytic conductor of f and non-trivial zeros of $L(s,f)$ denoted by $\frac{1}{2}+\gamma_f^{(j)}$.

The analytic conductor arises from the functional equation of the L-function, and near the central point the spacing between adjacent zeros is on the order of the reciprocal of its logarithm. This thus provides the natural scale to study the distance of zeros from the central point; on average it is around $1/\log c_f$ from one such zero to another; we are interested in how often the first or first few zeros are less than a fixed percentage of this quantity. In other words, how often is the first zero less than half the average spacing?

NOTE: Throughout this paper, whenever we talk about bounds on the location of zeros near the central point, we always mean relative to the average spacing; thus a bound of .25 means .25 times the reciprocal of the logarithm of the analytic conductor.

In order to get the optimal result possible, we usually choose a test function Φ with certain characteristics that will help us yield better bounds. We always choose Φ to be an even Schwartz function as zeros are distributed symmetrically on the critical line.[2] One main characteristic of the Schwartz test function is that the function along with all its derivatives decays faster than any polynomial as $|x|$ goes to infinity.

In order to execute the number theory sums that arise, we need the test function to have compact support for its Fourier transform (defined in 2.1). This is due to the fact that we need fairly restricted support for the Fourier transform in order to compute the n-level density. If we could have the test function concentrated near the central point and rapidly decaying, we would glean a significant amount of information on what is happening at the central point; for example, if we could take a delta spike then the only contribution would be from zeros at the central point. Sadly, just like the Heisenberg Uncertainty Principle that illustrates the trade-off between a particle's position and momentum, we cannot have both of the above conditions fulfilled

[2] Thus if we write our test function as the sum of an even and an odd function, the sum of the odd part over the zeros would vanish, so there is no need to include it.

simultaneously. A more concentrated test function means a larger support of its Fourier transform, and a more compact support of the Fourier transform results in less decay for the test function.

To investigate the behavior of the first (or first few) zeros above the central point, we impose additional requirements for the test function, with the conditions varying depending on what we are trying to prove.

For Theorem 1.8, we use the naive test function defined as follows.

Definition 1.4 (Naive Test Function). *For* $\operatorname{supp}(\widehat{\phi}_{\text{naive}}) \subset (-\sigma_n, \sigma_n)$, *set*

$$\phi_{\text{naive}}(x) \;=\; \left(\frac{\sin(\pi\sigma_n x)}{(\pi\sigma_n x)}\right)^2. \tag{1.11}$$

The Fourier transform is

$$\widehat{\phi}_{\text{naive}}(y) = \frac{1}{\sigma_n}\left(y - \frac{|y|}{\sigma_n}\right). \tag{1.12}$$

Since the naive test function is non-negative, it is useful when trying to get upper bounds as we can drop non-negative contributions of the density. Some previous studies such as [HM] attempt to find the optimal non-negative test functions for certain level densities; however, since the improvement in results is quite small, we are satisfied with just using the naive test function as it greatly simplifies the exposition, though with additional work slight improvements are possible.

For Theorem 1.6, we have a different set of characteristics from the test function ϕ. Contrary to being completely non-negative, we now only want it to be non-negative within a certain distance from the central point, and non-positive beyond (and $\phi(0) \neq 0$). Such a setup allows us to drop non-positive contributions to the density, thus getting a lower bound. To find such ϕ, we started off by constructing its Fourier transform since it has finite support (we will provide more details in section 3.1).

1.4. **Centered Moments.** The n-th centered moments are alternative, but essentially equivalent after some combinatorics, statistics to the n-level densities. Their expansions are better suited to obtain bounds (see [HM]). For the following theorem that was proven in [C–], we are only able to take the support up to 2, though the Random Matrix Theory conjectured that we can take larger support.

Theorem 1.5. *Let* $n \geq 2$ *and* $\operatorname{supp}(\phi) \subset (-\frac{\sigma}{n}, \frac{\sigma}{n})$, *where* $\sigma = 2$. *Define*

$$\sigma_\phi^2 \;:=\; 2\int_{-\infty}^{\infty} |y|\widehat{\phi}(y)^2 dy \tag{1.13}$$

and

$$\begin{aligned} R(m,i;\phi) \;:=\; & 2^{m-1}(-1)^{m+1}\sum_{l=0}^{i-1}(-1)^l\binom{m}{l} \\ & \left(-\frac{1}{2}\phi^m(0) + \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\widehat{\phi}(x_2)\cdots\widehat{\phi}(x_{l+1})\right. \\ & \left.\int_{-\infty}^{\infty}\phi^{m-l}(x_1)\frac{\sin(2\pi x_1(1+|x_2|+\cdots+|x_{l+1}|))}{2\pi x_1}dx_1\cdots dx_{l+1}\right) \end{aligned} \tag{1.14}$$

and

$$S(n,a,\phi) \;:=\; \sum_{l=0}^{\lfloor\frac{a-1}{2}\rfloor}\frac{n!}{(n-2l)!l!}R(n-2l,a-2l,\phi)\left(\frac{\sigma_\phi^2}{2}\right)^l. \tag{1.15}$$

By $\langle Q(f)\rangle_\pm$ *we mean the average of* $Q(f)$ *over all* f *in the family of even (odd) cuspidal newforms of level* N *for the positive (negative) sign (the number of such forms is proportional to* N*). Then*

$$\lim_{\substack{N\to\infty \\ N\text{prime}}}\left\langle\left(D(f;\phi) - \langle D(f;\phi)\rangle_\pm\right)^n\right\rangle_\pm \;=\; 1_{n\text{ even}}(n-1)!!\sigma_\phi^n \pm S(n,a;\phi), \tag{1.16}$$

for

$$1_{n \text{ even}}(n) \;=\; \begin{cases} 1 & \text{if } n = 2m \text{ is even} \\ 0 & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$$

1.5. **Main Results.** We obtain results on the number and location on low-lying zeros of the family of cuspidal newforms of level N through n-th centered moment. In particular, in Theorem 1.6, we give a lower bound $(-\omega, \omega)$ on the lowest first zero of the family through odd-centered moments. We further prove additional results in Theorem 1.8 for zeros near the central point and provide an upper bound on the percent of forms in the family with at least r number of zeros within a certain interval $(-\rho, \rho)$ near the central point.

Previous work has focused on using the 1-level density to obtain such bounds; the n-level densities and n^{th} centered moments are far more difficult to work with, leading to significantly harder integrals as well as some combinatorial challenges in removing the contributions of certain forms and isolating the desired results; however, as the support increases these results are significantly better than those from the 1-level, justifying the efforts. Instead of trying to bound the lowest zero of all forms, our question is different as we are searching for the lowest of the low and thus are able to get significantly better results.

Theorem 1.6. *Let $h(y)$ be an even function that is at least twice continuously differentiable. The support of h lies between -1 and 1, and it also monotonically decrease from 0 to 1. We define $f(y) = h(2yn/\sigma)$, $g(y) = (f*f)(y)$ (convolution of f with f). Let $\widehat{\phi}_\omega(y)$, the Fourier transform of $\phi_\omega(x)$, equal $g(y) + (2\pi\omega)^{-2}g''(y)$. Thus we have $\operatorname{supp}(\widehat{\phi}_\omega) \subset (-\sigma/n, \sigma/n)$, and $\phi_\omega(x)$ non-negative when $|x| \le \omega$ and non-positive when $|x| > \omega$. Then for an odd $n = 2m+1$, whenever ω satisfies this following inequality*

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y)dy\right)^n \;<\; 1_{n \text{ even}}(n-1)!!\sigma_{\phi_\omega}^n + S(n, a; \phi_\omega), \tag{1.17}$$

there exists at least one form with at least one normalized zero on the interval $(-\omega, \omega)$. In particular, if we denote $(-\omega_{\min}, \omega_{\min})$ to be the interval near the central point that contains at least one normalized zero, then the one-level density gives us an explicit expression that there is at least one zero in this interval whenever $\omega_{\min}$ satisfieds

$$\omega_{\min}(\sigma, h) \;>\; \left(-\frac{\sigma\int_0^1 h(u)^2\,du + \frac{\sigma^2}{4}\int_{-1}^1\int_0^{2/\sigma} h(u)h(v-u)\,dv\,du}{\frac{1}{\sigma}\int_0^1 h(u)h''(u)\,du + \frac{1}{4}\int_{-1}^1\int_0^{2/\sigma} h(u)h''(v-u)\,dv\,du}\right)^{-\frac{1}{2}} \pi^{-1}. \tag{1.18}$$

For the support $\sigma = 2$ (which is the largest known support if we only assume GRH, though we conjecture the result holds for all σ) and $h = \cos(\pi x/2)$ for $|x| \le 2$ and 0 otherwise (the optimum function to use for one-level density), the smallest first non-trivial zero of the family of even cuspidal newforms has imaginary part γ in the interval $-\gamma_{\min} \le \gamma \le \gamma_{\min}$, where $\gamma_{\min} \approx 0.25$.

While increasing the level n will give better results, if we can only use $\sigma = 2$ (currently the best known result under the assumption of just GRH) as we compare the bounds arising from using results from levels $1, 3$ and 5, we actually see worse results as the level increase. We prove that ω and σ are inversely proportional, which means that if we take a larger σ, ϕ_ω can be more focused around the central point, modeling the delta spike, and then the benefit of raising to the power n comes into play and yields smaller intervals. The Random Matrix Theory conjectures imply that we can take σ arbitrarily large (though as remarked results are proven in the number theory side, for sums over zeros, under GRH only for σ up to 2, though assuming additional conjectures allows one to increase slightly). Thus, if we use a σ greater than 2, we see that eventually as the support is large enough, the higher levels yield better results.

Remark 1.7. *We examined additional results from level 3 and 5. Unfortunately, higher levels involve difficult integrals to evaluate that do not have closed form evaluations, especially when we are using a complicated test function involving Fourier transforms and convolution. While we cannot use theoretical arguments or software packages such as Mathematica to evaluate the multi-dimensional integrals on the right-hand*

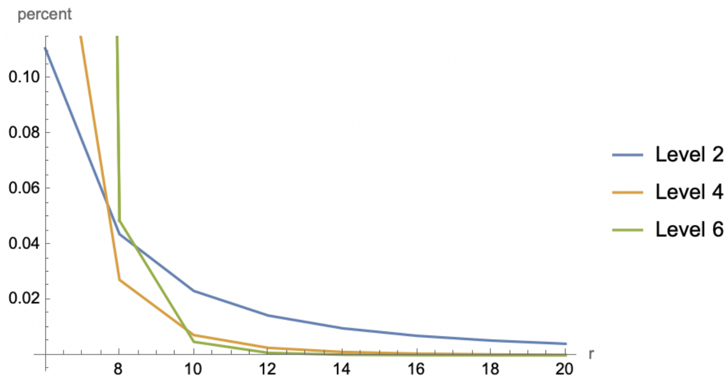

FIGURE 1. Percentage vs. number of zeros (for a fixed $\rho = .4$)

side, by using Riemann sums instead we can calculate the integrals accurately enough by choosing the step size Δx sufficiently small, and evaluate the integrals up to a small enough error for our purposes. We found that the result from the 3-level eventually is better than that from 1-level, for example when $\sigma = 7$, and the 5-level statistics beats both one and three levels when $\sigma = 6$. These results highlight the importance of deriving the theory on bounds arising from higher moments.

Previous investigations have solely been concerned with bounds on the first zero above the central point, obtaining either upper or lower bounds on its height. We introduce a new problem to the field, namely bounding the percentage of forms with a large number of zeros in a small window about the central point; there is tremendous freedom in assigning values to "large" and "small" above; in brief, we obtain excellent bounds that the probability is very low of many more zeros in a small window than expected.

Specifically, we bound the percentage of forms in the family of cuspidal newforms $\mathcal{F}_N$ with r zeros on the interval $(-\rho, \rho)$. We are aware of no similar results in the literature, as to date all previous work has been focused on just one zero.

Theorem 1.8. *Let $P_{r,\rho}(\mathcal{F})$ denote the limit when N tends infinity of the percent of forms in $\mathcal{F}_N$ that have at least r normalized zeros on the interval $(-\rho, \rho)$. For $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$ and $n = 2m$ is even,*

$$P_{r,\rho}(\mathcal{F}) \ \leq \ \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathcal{F}))^n}, \tag{1.19}$$

where ϕ is an even, non-negative Schwartz test function.

Previous studies [DM], which bounded the order of vanishing through n-level densities and n-th centered moments, observed an interesting phenomenon when one increased the level. While it is predicted that higher levels would yield better bounds, one only sees the improvement for higher order vanishing (note this is similar to the effect we observed above in where there must be at least one form with at least one zero sufficiently close). In fact, better bounds for small ranks are produced through lower levels. We see a similar pattern in our bounds resulting from Theorem 1.8.

If we plot out the percentage vs. number of zeros r graph for a fixed interval $(-\rho, \rho)$ (see Figure 1), we see that although the graph of higher levels starts off a lot higher than lower levels when r is small, they decrease faster and eventually gives better results as r increases. For instance, if we compare the bound from level 2 to level 6 for a fixed $\rho = .4$, we see that when $r = 6$, the 2-level gives a bound of 0.111085 (i.e., at most approximately 11% have a zero within 40% of the average spacing near the central point) while the 6-level yields 1.58572 (which is a useless bound as it exceeds a hundred percent!). However, when we look at when $r = 20$, we see a significantly better bound from level 6 that gives 0.0000121805 compared to level

2 that yields 0.0042038. Thus the higher levels yield greater decay in the probability of the unlikely case of a large number of zeros very close to the central point, far better than the results from the 2-level density.

We also see that for a fixed number of zeros, the higher level densities give better bounds for a bigger interval $(-\rho, \rho)$ near the central point (see tables in section 4.1). For instance, for a fixed $r = 8$, the 4-level density gives a bound of 0.0331395 at $\rho = .5$, smaller than the 6-level which yields 0.0534808. However, when we increase ρ to be .9, we have the level six giving a better bound of 0.847282 than 2.14456 from level four.

2. Preliminaries

First, we establish the necessary notation and present the required preliminary results, starting with discussing certain properties of the Fourier Transform. Often, it's more straightforward to transform a problem related to functions into one concerning their Fourier transforms and then revert to the original function at the conclusion. This approach is beneficial because the Fourier transform of a convolution (which, as we'll discuss later, is an integral) equates to the multiplication of the individual Fourier transforms. This method turns the complex process of integration into the simpler task of multiplication. However, this comes with the trade-off of needing to compute the inverse Fourier transform later on. For detailed proofs of the Fourier transform properties see Appendix A and [SS].

Definition 2.1 (Fourier Transform)**.** *The Fourier transform of a function $\phi(x)$ is defined by*

$$\mathfrak{F}(\phi(x)) \ := \ \widehat{\phi}(y) \ = \ \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i x y} dx. \tag{2.1}$$

A simple calculation yields the following (see proof in Appendix A).

Lemma 2.2. *The scaling property of the Fourier transform with a constant c is:*

$$\mathfrak{F}(\phi(cx)) \ = \ \frac{1}{|c|} \cdot \widehat{\phi}\left(\frac{y}{c}\right) \tag{2.2}$$

Definition 2.3 (Convolution)**.** *The convolution of $a(x)$ and $b(x)$ is defined as*

$$(a * b)(x) \ := \ \int_{-\infty}^{\infty} a(t) b(x - t) dt. \tag{2.3}$$

Note if a and b are functions supported in $(-s, s)$ then $a * b$ is supported in $(-2s, 2s)$.

The following theorem converts convolutions to easier quantities to study (see proof in Appendix A.

Theorem 2.4 (Convolution Theorem)**.** *For functions $a(x)$ and $b(x)$,*

$$\mathfrak{F}(a * b) \ = \ \mathfrak{F}(a)\mathfrak{F}(b). \tag{2.4}$$

Notice that if a and b are supported in $(-\eta, \eta)$ that the convolution is supported up to $(-2\eta, 2\eta)$.

The next lemma shows a nice interplay between convolution and differentiation; see Appendix A for a proof.

Lemma 2.5. *The double derivative of the convolution between two functions $a(x)$ and $b(x)$ is*

$$(a * b)''(x) \ = \ a(x) * b''(x). \tag{2.5}$$

We introduce the Fourier transform pair of the sinc function and the rectangular function. We first define both of them.

Definition 2.6 (Sinc function)**.** *The normalized sinc function is defined as*

$$\operatorname{sinc} x \ = \ \begin{cases} \frac{\sin \pi x}{\pi x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \tag{2.6}$$

Lemma 2.7 (Rectangular function)**.** *The rectangular function, also known as the normalized boxcar function, is defined as*

$$\operatorname{rect}\left(\frac{x}{a}\right) \;=\; \begin{cases} 0 & \text{if } |x| > \frac{a}{2} \\ \frac{1}{2} & \text{if } |x| = \frac{a}{2} \\ 1 & \text{if } |x| < \frac{a}{2}. \end{cases} \tag{2.7}$$

Lemma 2.8. *The sinc function and the rectangular function are the Fourier transform of each other, which means that*

$$\mathfrak{F}(\operatorname{sinc}(x)) \;=\; \int_{-\infty}^{\infty} \operatorname{sinc}(x) e^{-2\pi ixy} dx \;=\; \operatorname{rect}(x) \tag{2.8}$$

and

$$\mathfrak{F}\left(\operatorname{rect}\left(\frac{x}{a}\right)\right) \;=\; \int_{-\infty}^{\infty} \operatorname{rect}(\frac{x}{a}) e^{-2\pi ixy} dx \;=\; a\operatorname{sinc}(ax). \tag{2.9}$$

Plancherel's theorem is sometimes useful when integrating the integral of a product of two functions, converting the the integral of a product to the integral of the product of their Fourier transforms. We use this on the space of $L^2(\mathbb{R})$, which are functions whose integral of the square of their absolute value is finite; as our test functions are bounded and decay rapidly, they are all in this space.

Theorem 2.9 (Plancherel theorem)**.** *For two $L^2(\mathbb{R})$ functions $a(x)$ and $b(x)$,*

$$\int_{-\infty}^{\infty} a(x)\overline{b(x)}dx \;=\; \int_{-\infty}^{\infty} \mathfrak{F}((a(x))\overline{\mathfrak{F}(b(x))}dx. \tag{2.10}$$

In our analysis we have equalities where the left hand side (LHS) involves sums over zeros, while the right hand side (RHS) are sums of integrals whose integrands are involved products of quantities related to our test functions and the n-level density kernels. As these integrals are impossible to evaluate in closed form for our test functions, we use Riemann Sum to approximate the integrals; by choosing the step size sufficiently small we can ensure that the required number of digits of our answer are correct.

Theorem 2.10 (Riemann Sum)**.** *Let $f : [a,b] \to \mathbb{R}$ be a function defined on a closed interval $[a,b]$ and $a, b \in \mathbb{R}$. Let $P = (x_0, x_1, \dots, x_n)$ be the partition of $[a,b]$ such that*

$$a \;:=\; x_0 \;<\; x_1 \;<\; \cdots \;<\; x_n \;=:\; b. \tag{2.11}$$

The Riemann sum S of f over $[a,b]$ with partition P is defined as

$$S \;:=\; \sum_{k=1}^{n} f(x_k^*)\Delta x_k, \tag{2.12}$$

where $\Delta x_k := x_k - x_{k-1}$ and $f(x_k^*) \in [x_{k-1}, x_k]$. The sum converges to the integral of f over $[a,b]$ as $\max_k \Delta x_k$ approaches zero.

The quantities on the RHS often involve combinatorial factors, including the double factorial.

Definition 2.11 (Double Factorial)**.** *The double factorial of n is the product of all integers up to n that shares the same parity (odd or even):*

$$n!! \;:=\; \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n - 2k) \;=\; n(n-2)(n-4)(n-6)\cdots. \tag{2.13}$$

Definition 2.12. *Let $\mathcal{F}_{N,r}^{(\rho)}$ denote the family of forms in $\mathcal{F}_N$ that have at least r zeros within a distance ρ. For a fixed r, as N tends to infinity through the primes, define $P_{r,\rho}(\mathcal{F})$ to be the limit of the percent of forms in $\mathcal{F}_N$ that have at least r zeros within a distance ρ:*

$$P_{r,\rho}(\mathcal{F}) \;:=\; \lim_{N\to\infty} \frac{|\mathcal{F}_{N,r}^{(\rho)}|}{|\mathcal{F}_N|}. \tag{2.14}$$

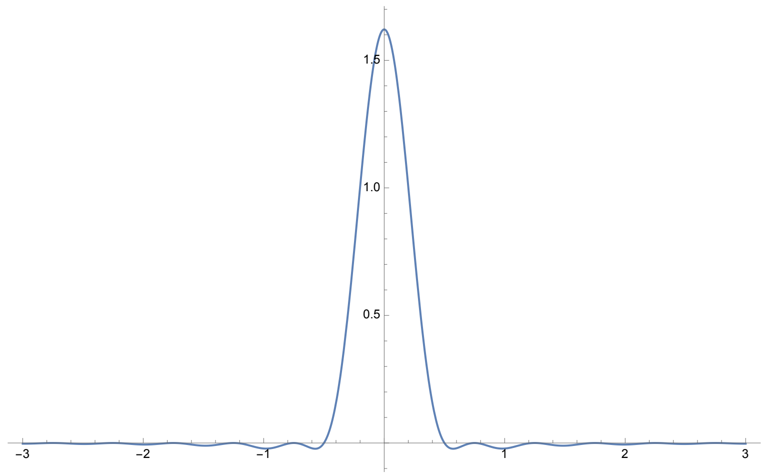

FIGURE 2. Plot of $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$, for $h = \cos\left(\frac{\pi y}{2}\right)$, $\sigma = 2$, $\omega = .5$, and $n = 1$.

3. PROOF OF THEOREM 1.6

3.1. **Construction of test function.** We first outline the construction of the test function ϕ_ω that meets the required conditions (we adopt a similar process from [GM] and [HR]).

We start by building the Fourier transform $\widehat{\phi}(y)$ that has a finite support $(-\frac{\sigma}{n}, \frac{\sigma}{n})$ since it is not easy to find a function that satisfies all the conditions for ϕ directly. Let $h(y)$ be an even function that is at least twice continuously differentiable. The support of h lies between -1 and 1, and it also monotonically decrease from 0 to 1. For a fixed ω and σ, let $f(y) = h(\frac{2yn}{\sigma})$ and $g(y) = (f * f)(y)$ (the convolution of f with itself (2.3)), and let $\widehat{\phi_\omega}(y)$ equal $g(y) + (2\pi\omega)^{-2}g''(y)$ (where $\widehat{\phi_\omega}(y)$ is the Fourier transform of $\phi_\omega(x)$).

We show that $\operatorname{supp}\widehat{\phi_\omega}(y) \subset (-\sigma/n, \sigma/n)$. Notice, since $\operatorname{supp}(f) \subset (-\sigma/2n, \sigma/2n)$ and $g = f * f$, $\operatorname{supp}(g) \subset (-\sigma/n, \sigma/n)$. Since $\widehat{\phi_\omega}(y) = g(y) + (2\pi\omega)^{-2}g''(y)$ and the support of g'' is contained in that of g, the support of $\widehat{\phi_\omega}(y)$ is contained in $(-\frac{\sigma}{n}, \frac{\sigma}{n})$, corresponding to what we want.

We find $\phi_\omega(x)$ by taking the Fourier transform of $\widehat{\phi_\omega}(y)$[3]. Notice the Fourier transform of $g''(y)$ is $-(2\pi y)\widehat{g}(y)$ and $g'' = f * f''$. Combining the above, we have the Fourier transform of $\widehat{\phi_\omega}(y) = g(y) + (2\pi\omega)^{-2}g''(y)$ is $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$, shown in Figure 2.

Remark 3.1. *Following the framework of constructing the test function above, one can possibly adopt a more elaborate construction, such as adding a quartic term in ϕ, to obtain better test functions that yields better results.*

3.2. **Proof of Theorem 1.6.** We use odd centered moments to obtain a lower bound on the number of zeros within a small distance ω from the central point. Then we prove our statement by showing a contradiction when assuming there are no zeros in the interval $(-\omega, \omega)$.

As remarked in the introduction, there is a long history of results on the n-level densities and the combinatorially easier centered moments. The first results are due to Iwaniec-Luo-Sarnak [ILS], who computed the 1-level density with support up to 2. Their result was later extended by Hughes-Miller [HM] to the n-level densities and centered moments with support up to $1/(n-1)$. Recently there was a breakthrough by Cohen et. al. [C–], who was finally able to handle the combinatorics and prove the n-level densities and centered moments agree with random matrix theory for support up to $2/n$. We restate Theorem 1.5 for convenience.

---
[3]One of the properties of Fourier transform is $\mathfrak{F}(\mathfrak{F}(\phi(x))) = \phi(-x)$. In this case, since ϕ is even, we are able to find the original function using this property directly.

Theorem 3.2. *Let $n \geq 2$ and $\text{supp}(\phi) \subset (-\frac{\sigma}{n}, \frac{\sigma}{n})$, where $\sigma = 2$. Define*

$$\sigma_\phi^2 \ := \ 2\int_{-\infty}^{\infty} |y|\widehat{\phi}(y)^2 dy \tag{3.1}$$

and

$$\begin{aligned} R(m,i;\phi) \ :=& \ 2^{m-1}(-1)^{m+1}\sum_{l=0}^{i-1}(-1)^l\binom{m}{l} \\ & \Bigg(-\frac{1}{2}\phi^m(0) + \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\widehat{\phi}(x_2)\cdots\widehat{\phi}(x_{l+1}) \\ & \int_{-\infty}^{\infty}\phi^{m-l}(x_1)\frac{\sin(2\pi x_1(1+|x_2|+\cdots+|x_{l+1}|))}{2\pi x_1}dx_1\cdots dx_{l+1}\Bigg) \end{aligned} \tag{3.2}$$

and

$$S(n,a,\phi) \ := \ \sum_{l=0}^{\lfloor\frac{a-1}{2}\rfloor}\frac{n!}{(n-2l)!l!}R(n-2l,a-2l,\phi)\left(\frac{\sigma_\phi^2}{2}\right)^l. \tag{3.3}$$

Then

$$\lim_{\substack{N\to\infty \\ N \text{prime}}}\left\langle\left(D(f;\phi)-\langle D(f;\phi)\rangle_\pm\right)^n\right\rangle_\pm \ = \ 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n,a;\phi). \tag{3.4}$$

Corollary 3.3. *We can write* (1.16) *as*

$$\lim_{\substack{N\to\infty \\ N \text{prime}}}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\left(\sum_j \phi(\gamma_{f,j}c_n)-\mu(\phi,\mathcal{F})\right)^n \ = \ 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n,a;\phi), \tag{3.5}$$

where $\mu(\phi,\mathcal{F})$ is the mean of the family of cuspidal newforms.

Proof. We are able to see that the difference between the mean of the family and the mean at the N level is negligible through the application of Binomial Theorem and the Cauchy-Schwartz inequality. Thus to make the calculation easier, we are able to replace $\mu(\phi,\mathcal{F}_N)$ with $\mu(\phi,\mathcal{F})$. For a more complete proof see [DM]. □

Theorem 3.4 (See [ILS, HM])**.** *The mean over SO(even) is*

$$\mu(\phi,\mathcal{F}) \ := \ \widehat{\phi}(0) + \frac{1}{2}\int_{-\infty}^{\infty}\widehat{\phi}(y)dy. \tag{3.6}$$

We restate our first main result, Theorem 1.6, for convenience, and then give the proof.

Theorem 3.5. *For n odd, whenever ω satisfies the inequality*

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n}\widehat{\phi}_\omega(y)dy\right)^n \ < \ S(n,a;\phi_\omega), \tag{3.7}$$

there exists at least one form with one normalized zero on the interval $(-\omega,\omega)$.

Proof. As remarked earlier we focus only the family of even forms; there are trivial modifications if one studies the odd forms (since all forms with an odd functional equation are guaranteed a zero at the central point, we would need to remove that contribution).

We start with equation (3.5). As ϕ_ω is non-negative when $|x| \leq \omega$ and non-positive when $|x| > \omega$, the contribution of the scaled zeroes to the centered moment when $|x| \geq \omega$ is non-positive. Since the power is odd, if we drop those non-positive contributions, it will not increase the left-hand side. Thus the centered moment gives a lower bound

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_{|\gamma_{f,j}|\le\omega} \phi_\omega(\gamma_{f,j} c_n) - \mu(\phi_\omega, \mathcal{F}) \right)^n \ \geq\ S(n,a;\phi_\omega). \tag{3.8}$$

Assume no forms have a zero on the interval $(-\omega, \omega)$. Then the contributions of the scaled zeroes to the centered moment when $|x| \le \omega$ equals 0. We get

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} (-\mu(\phi_\omega, \mathcal{F}))^n \ \geq\ S(n,a;\phi_\omega). \tag{3.9}$$

Since the mean is not dependent on f, we can extract it from the summation:

$$(-\mu(\phi_\omega, \mathcal{F}))^n \lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 \ \geq\ S(n,a;\phi_\omega). \tag{3.10}$$

Notice $\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 = \lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{|\mathcal{F}_N|}{|\mathcal{F}_N|} = 1$, so we get

$$(-\mu(\phi_\omega, \mathcal{F}))^n \ \geq\ S(n,a;\phi_\omega). \tag{3.11}$$

Because of the compact support of $\widehat{\phi}_\omega$, by Theorem 3.4, this is

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2} \int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y) dy \right)^n \ \geq\ S(n,a;\phi_\omega). \tag{3.12}$$

Thus, if we choose ω that fulfills the following inequality

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2} \int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y) dy \right)^n \ <\ S(n,a;\phi_\omega), \tag{3.13}$$

we get a contradiction, indicating there has to be at least one form with one normalized zero on the interval $(-\omega, \omega)$. □

Remark 3.6. *The theorem above improves itself as the zeros are distributed symmetrically along the critical line, and thus if $1/2 + i\gamma$ is a zero so too is $1/2 - i\gamma$. In fact, there are at least two zeros in such a window.*

3.3. **Explicit bounds on first zero.** We examine the inequality in 3.13 and give an explicit formula for the one-level calculation of the bound ω.

Theorem 3.7. *The first zero of the family of cuspidal newforms exists on the interval $(\omega_{\min}, \omega_{\min})$, where*

$$\omega_{\min}(\sigma, h) \ >\ \left(-\frac{\sigma \int_0^1 h(u)^2\, du + \frac{\sigma^2}{4} \int_{-1}^1 \int_0^{2/\sigma} h(u)h(v-u)\, dv\, du}{\frac{1}{\sigma} \int_0^1 h(u)h''(u)\, du + \frac{1}{4} \int_{-1}^1 \int_0^{2/\sigma} h(u)h''(v-u)\, dv\, du} \right)^{-\frac{1}{2}} \pi^{-1}. \tag{3.14}$$

Proof. Note that $\int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y) dy = \phi_\omega(0)$. Thus for $n = 1$, we can rewrite the inequality in (3.13) as

$$-\widehat{\phi}_\omega(0) - \frac{1}{2}\phi_\omega(0) \ <\ S(1,1;\phi_\omega). \tag{3.15}$$

By Theorem 1.5, we can re-write the RHS as

$$S(1,1;\phi_\omega) \ =\ R(1,1;\phi_\omega) \ =\ -\frac{1}{2}\phi_\omega(0) + \int_{-\infty}^{\infty} \phi_\omega(x) \frac{\sin(2\pi x)}{2\pi x}\, dx. \tag{3.16}$$

The $-1/2\phi_\omega(0)$ term cancels out with left-hand side, so we focus on trying to simplify the second term. Notice that directly integrating with the $\sin$ function is hard, so instead we use the Plancherel theorem 2.9

and rewrite the integrand as the product of their Fourier transforms. The Fourier transform of $\phi_\omega(x)$ is $\widehat{\phi}_\omega(y)$. In order to properly integrate $\sin(2\pi x)/2\pi x$, we can take it as the sinc function defined in 2.6. Since the Fourier transform of the sinc function is the rectangular function, with some re-scaling by 2.2 we get

$$\int_{-\infty}^{\infty} \phi_\omega(x) \frac{\sin(2\pi x)}{2\pi x}\, dx \;=\; 2\int_{0}^{\infty} \widehat{\phi}_\omega(y) \frac{1}{2} \operatorname{rect}\left(\frac{y}{2}\right)\, dy. \tag{3.17}$$

Notice the rectangular function $\operatorname{rect}\left(\frac{y}{2}\right)$ equals 1 from -1 to 1 and 0 everywhere else, thus it is safe to rewrite the above integral as $\int_0^1 \widehat{\phi}_\omega(y)dy$, integrating up to 1. Thus the RHS is simplified to $-\frac{1}{2}\phi_\omega(0) + \int_0^1 \widehat{\phi}_\omega(y)dy$. Combining with what we have for LHS from (3.15), we get that the bound for the smallest first zero is when

$$-\widehat{\phi}_\omega(0) \;<\; \int_0^1 \widehat{\phi}_\omega(y)\, dy. \tag{3.18}$$

We now simplify $\widehat{\phi}_\omega(0)$ in terms of an arbitrary h that fulfills the requirements. Expanding the construction of $\widehat{\phi}_\omega$,

$$\widehat{\phi}_\omega(0) \;=\; g(0) + (2\pi\omega)^{-2} g''(0). \tag{3.19}$$

Through the construction of g and by taking advantage of the evenness of h and f, we get

$$g(0) = \int_{-\infty}^{\infty} f(t)f(0-t)dt = 2\int_0^{\sigma/2} h\left(\frac{2tn}{\sigma}\right)^2 dt = \sigma \int_0^1 h(u)^2 du. \tag{3.20}$$

As $g = f * f$ implies $g'' = f * f''$ we find

$$\begin{aligned}
g''(0) \;&=\; \int_{-\sigma/2}^{\sigma/2} f(t) f''(0-t)\, dt \\
&=\; 2\int_0^{\sigma/2} f(t) f''(t)\, dt \\
&=\; 2\int_0^{\sigma/2} h\left(\frac{2t}{\sigma}\right) \frac{4}{\sigma^2} h''\left(\frac{2t}{\sigma}\right)\, dt \\
&=\; \frac{4}{\sigma}\int_0^1 h(u) h''(u)\, du.
\end{aligned} \tag{3.21}$$

Combining all of the above, we get

$$-\widehat{\phi}_\omega(0) \;=\; -\sigma\int_0^1 h(u)^2\, du - (2\pi\omega)^{-2}\frac{4}{\sigma}\int_0^1 h(u)h''(u)\, du. \tag{3.22}$$

We now simplify the RHS of (3.18). By the expansion of $\widehat{\phi}_\omega$,

$$\int_0^1 \widehat{\phi}_\omega(y)\, dy \;=\; \int_0^1 g(y)\, dy + (2\pi\omega)^{-2}\int_0^1 g''(y)\, dy. \tag{3.23}$$

To rewrite the integrals in terms of h, we outline the procedure to simplify the first term from above, and the second term follows analogously[4].

[4] The Fubini-Tonelli theorem states that we can change the order of integration for a two dimensional integral if it yields a finite values when the integrand is replaced by its absolute value. Since the function we are using is finitely supported, using this theorem is justified.

$$
\begin{aligned}
\int_0^1 g(y)\,dy &= \int_0^1 \int_{-\infty}^{\infty} f(t)f(y-t)\,dt\,dy \\
&= \int_0^1 \int_{-\sigma/2}^{\sigma/2} h\left(\frac{2t}{\sigma}\right) h\left(\frac{2(y-t)}{\sigma}\right)\,dt\,dy \\
&= \frac{\sigma}{2}\int_0^1 \int_{-1}^{1} h(u)h\left(\frac{2y}{\sigma}-u\right)\,du\,dy \\
&= \frac{\sigma}{2}\int_{-1}^1 \int_{0}^{1} h(u)h\left(\frac{2y}{\sigma}-u\right)\,dy\,du \\
&= \frac{\sigma^2}{4}\int_{-1}^1 \int_{0}^{2/\sigma} h(u)h(v-u)\,dv\,du. \qquad (3.24)
\end{aligned}
$$

We go through the same procedure to simplify the second term. Eventually, the right-hand side becomes

$$
\int_0^1 \widehat{\phi}_\omega(y)\,dy \;=\; \frac{\sigma^2}{4}\int_{-1}^1 \int_{0}^{2/\sigma} h(u)h(v-u)\,dv\,du + (2\pi\omega)^{-2}\int_{-1}^1 \int_{0}^{2/\sigma} h(u)h''(v-u)\,dv\,du. \qquad (3.25)
$$

Putting the simplified versions of two sides of (3.18) together and isolate ω, we get the expression for the minimum ω:

$$
\omega \;>\; \left(-\frac{\sigma\int_0^1 h(u)^2\,du + \frac{\sigma^2}{4}\int_{-1}^1\int_0^{2/\sigma} h(u)h(v-u)\,dv\,du}{\frac{1}{\sigma}\int_0^1 h(u)h''(u)\,du + \frac{1}{4}\int_{-1}^1\int_0^{2/\sigma} h(u)h''(v-u)\,dv\,du}\right)^{-\frac{1}{2}} \pi^{-1}. \qquad (3.26)
$$

□

Remark 3.8. *Since number theory has only been able to compute these quantities for support σ up to 2 (under the assumption of GRH), we get $\omega_{\min}(2,h) \approx 0.25$ for $h = \cos(\pi y/2)$ is the optimum function that yields the smallest bound. We also looked at results for level 3 and 5 by directly integrating both sides of the inequality, hoping to see better bounds than the one-level. Since the right-hand side involves multi-dimensional integral integrating over a very complicated ϕ_ω function (involving Fourier transforms and convolutions), we evaluate it using Riemann sums, see (2.12). Although it cannot give exact values, the approximation is sufficiently close as we use a very small Δx. For our evaluation, we chose to use the simple function $h(y) = 1 - y^2$ to construct our test function ϕ_ω. This is because the complex nature of the right-hand side makes evaluation extremely slow, and since optimizing the function have little improvement in the final results, we are satisfied with using the chosen h for now (though ongoing studies will focus on finding the optimal function to use). However, although we expect higher levels to yield better bounds for support sufficiently large, that is not the case here. As we increase the level, the bound gets worse for support $\sigma = 2$: for $n = 3$, $\omega \approx .34$, and for $n = 5$, $\omega \approx .85$.*

4. Proof of Theorem 1.8

We obtain an upper bound through even centered moments and using the naive test function ϕ defined in (1.11).

Theorem 4.1. *Let $P_{r,\rho}(\mathcal{F})$ denote the percent of forms in the family that has at least r zeros on the interval $(-\rho,\rho)$ near the central point. For an even n and $r \geq \mu(\phi,\mathcal{F})/\phi(\rho)$,*

$$
P_{r,\rho}(\mathcal{F}) \;\leq\; \frac{1_{n\text{ even}}(n-1)!!\sigma_\phi^n + S(n,a;\phi)}{(r\phi(\rho)-\mu(\phi,\mathcal{F}))^n}. \qquad (4.1)
$$

Proof. We first rewrite (3.5) as

$$\lim_{\substack{N\to\infty\\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_{|\gamma_{f,j}|\le\rho} \phi(\gamma_{f,j}c_n) + T_f(\phi) - \mu(\phi,\mathcal{F}) \right)^n \ = \ 1_{n\text{ even}}(n-1)!!\sigma_\phi^n + S(n,a;\phi), \quad (4.2)$$

where $T_f(\phi)$ is the contribution from the scaled $\phi(\gamma_{f,j}c_n)$ when $|\gamma_{f,j}| > \rho$.

We look at forms with at least r zeros within a distance ρ away from the central point, and denote the set of these forms by $\mathcal{F}_{N,r}^{(\rho)}$. Since n is even, by dropping all the form with less than r zeros in the interval $(-\rho,\rho)$, we are dropping a non-negative sum. Since doing so cannot increase the size of the left-hand side, we get the following upper bound:

$$\lim_{\substack{N\to\infty\\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}^{(\rho)}} \left(\sum_{|\gamma_{f,j}|\le\rho} \phi(\gamma_{f,j}c_n) + T_f(\phi) - \mu(\phi,\mathcal{F}) \right)^n \ \le \ 1_{n\text{ even}}(n-1)!!\sigma_\phi^n + S(n,a;\phi). \quad (4.3)$$

We then replace the summation of $\phi(\gamma_{f,j}c_n)$ with $r\phi(\rho)$, only counting the contribution of r zeros. This involves several steps, and importantly none of these lead to an increase in the LHS. This is because ϕ is non-negative, n is even, and since ϕ is decreasing from $(0,\rho)$, if we replace the contribution of the test function at the zeros by $\phi(\rho)$ we do not increase the sum. It is important to note that r is an even number, since by symmetry, if $1/2+i\gamma$ is a zero, so too is $1/2-i\gamma$.

We also want to be able to drop the term $T_f(\phi)$ and not increase left-hand side. However, although ϕ is non-negative, this if not the case if the first two terms in the parenthesis is smaller than the absolute value of the mean. Thus we restrict r to be greater than or equal to $\mu(\phi,\mathcal{F})/\phi(\rho)$, and further decrease the LHS by dropping $T_f(\phi)$:

$$\lim_{\substack{N\to\infty\\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}^{(\rho)}} (r\phi(\rho) - \mu(\phi,\mathcal{F}))^n \ \le \ 1_{n\text{ even}}(n-1)!!\sigma_\phi^n + S(n,a;\phi). \quad (4.4)$$

We move the inside the parenthesis out of the summation since the term inside does not depend on form f,

$$(r\phi(\rho) - \mu(\phi,\mathcal{F}))^n \lim_{\substack{N\to\infty\\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}^{(\rho)}} 1 \ \le \ 1_{n\text{ even}}(n-1)!!\sigma_\phi^n + S(n,a;\phi). \quad (4.5)$$

Notice, by (2.14), $\lim_{\substack{N\to\infty\\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_{N,r}^{(\rho)}} 1 = \lim_{\substack{N\to\infty\\ N\text{prime}}} \frac{|\mathcal{F}_{N,r}^{(\rho)}|}{|\mathcal{F}_N|} = P_{r,\rho}(\mathcal{F})$. We isolate the percentage and get

$$P_{r,\rho}(\mathcal{F}) \ \le \ \frac{1_{n\text{ even}}(n-1)!!\sigma_\phi^n + S(n,a;\phi)}{(r\phi(\rho) - \mu(\phi,\mathcal{F}))^n}. \quad (4.6)$$

□

4.1. **Explicit bounds for percentage.** We list some data of the explicit bounds calculated using Theorem 1.8. Notice that all number of zeros are even by symmetry.

As shown in both Table 1 and 2, we see the pattern that for bounding a small number of zeros, the 2-level does a significantly better job than the 4 or 6 level. In fact, both bounds for the 4 and 6 level when $r = 4$ are useless as they already exceed a hundred percent! Moreover, level four and six cannot even produce bounds for $r = 1$ ecause of the restriction of $r \ge \mu(\phi,\mathcal{F})/\phi(\rho)$ in our theorem. However, as we look at a larger number of zeros in the distance, the 4-level gradually gives a better bound than the 2-level. Eventually the 6-level beats both the 2-level and the 4-level in giving the smallest bound.

TABLE 1. Probability of forms having a given number of normalized zeros 0.2 away from the central point, using the naive test function with a support of $2/n$. "N/A" means that the level cannot give the bound because of the restriction in Theorem 1.8

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Number of zeros	2-level	4-level	6-level
2	6.651738	N/A	N/A
4	0.104108	2.370419	$2.147231 \cdot 10^{4}$
6	0.029617	0.069998	0.809927
8	0.013769	0.011079	0.022517
10	0.007924	0.003152	0.002427
12	0.005142	0.001213	$4.79813 \cdot 10^{-4}$
14	0.003605	$5.612212 \cdot 10^{-4}$	$1.340970 \cdot 10^{-4}$
16	0.002666	$2.942389 \cdot 10^{-4}$	$4.688515 \cdot 10^{-5}$
18	0.002052	$1.687747 \cdot 10^{-4}$	$1.917773 \cdot 10^{-5}$
20	0.001627	$1.036100 \cdot 10^{-4}$	$8.809943 \cdot 10^{-6}$

TABLE 2. Probability of forms having a given number of normalized zeros 0.4 away from the central point, using the naive test function with a support of $2/n$.

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Number of zeros	2-level	4-level	6-level
4	0.665694	8.334733	$1.744392 \cdot 10^{3}$
6	0.111085	0.145883	1.585718
8	0.043857	0.020351	0.036592
10	0.023310	0.005469	0.003680
12	0.014430	0.002038	0.000702
14	0.009804	0.000924	0.000192
16	0.007093	0.000475	$6.606784 \cdot 10^{-5}$
18	0.005369	0.000271	$2.673289 \cdot 10^{-5}$
20	0.004204	0.000165	$1.218053 \cdot 10^{-5}$

TABLE 3. Probability of forms having a given number of normalized zeros 0.8 away from the central point, using the naive test function with a support of $2/n$. "N/A" means that the level cannot give the bound because of the restriction in Theorem 1.8.

Number of zeros	2-level	4-level	6-level
6	N/A	10.849910	48.154279
16	N/A	0.004235	$2.83230 \cdot 10^{-4}$
26	N/A	$3.541901 \cdot 10^{-4}$	$6.716802 \cdot 10^{-6}$
28	420.045063	$2.486819 \cdot 10^{-4}$	$3.943864 \cdot 10^{-6}$
30	20.991406	$1.796948 \cdot 10^{-4}$	$2.418466 \cdot 10^{-6}$
32	6.651738	$1.330555 \cdot 10^{-4}$	$1.538761 \cdot 10^{-6}$
34	3.220871	$1.006126 \cdot 10^{-4}$	$1.010576 \cdot 10^{-6}$

We also see the pattern that for the same number of zeros, the chance of them existing in a smaller range $(-\rho, \rho)$ is much lower than in a slightly larger interval. For instance, the probability of having 8 zeros on the interval $(-.2, .2)$ is approximately 0.013769, while for the interval $(-.2, .2)$ it is much larger, about 0.043857 (for level two).

Higher level is also better at bounding zeros within a bigger distance from the central point. Because of the restriction of $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$, lower levels are simply incapable of bounding small number of zeros! For example, in Table 3 when we fixed a bigger $\rho = .8$, the 2-level can only bound the number of zeros $r > 27$, while the 4-level and 6-level are perfectly fine with r as small as 6. We also see that the 6-level producing much better bounds than smaller levels in Table 3. Thus, even though the higher levels are much more difficult to calculate, the tradeoff is worthwhile as we see its significance when we increase the interval $(-\rho, \rho)$ as well as the number of zeros r.

5. Future work

In Theorem 1.6 we focused only on the even forms of the family, as the odd forms are guaranteed to have one zero on the central point. Our result can be extended to odd forms by simply adding that contribution. It is important to note that this is a paper on functional analysis, where we use techniques in analysis as inputs to compute results for number theory quantities.

There are several directions one can take to improve the results in this paper. One is to optimize the test function. Though the optimum test function for one-level density have already been proven in previous studies, optimizing test function for higher level densities have not. Building off of the framework of the construction of the test function in Theorem 1.6, it is possible to alter it by adding a quartic term, for instance, to see improvement on bounds for higher levels. However, previous work highlights that the greatest gains come from increasing the support and using higher n-level, which is what we have focused on in this paper. At the cost of some additional standard computations, one can easily add additional terms and obtain similar integrals to approximate; initial explorations indicated that the gain was minimal, and thus we focused on the significant improvements that arise with higher level densities and moments, justifying the additional work required.

For higher levels, one can also try to simplify the limit calculation through complex analysis, making evaluation easier for explicit bounds.

Appendix A.

We give the proofs for some of the basic properties of Fourier transform used in this paper. For convenience we restate the theorems below.

Theorem A.1. *(Also Theorem 2.4) For functions $a(x)$ and $b(x)$,*

$$\mathfrak{F}(a * b) \ = \ \mathfrak{F}(a)\mathfrak{F}(b). \tag{A.1}$$

Proof.

$$\begin{aligned}
\mathfrak{F}((a*b)(x)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t)b(x-t)e^{-2\pi ixy}\,dt\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t)b(x-t)e^{-2\pi ixy}\,dx\,dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t)b(u)e^{-2\pi i(u+t)y}\,dx\,dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t)b(u)e^{-2\pi iuy}e^{-2\pi ity}\,dx\,dt \\
&= \int_{-\infty}^{\infty} a(t)e^{-2\pi ity}\int_{-\infty}^{\infty} b(u)e^{-2\pi iuy}\,dx\,dt \\
&= \int_{-\infty}^{\infty} a(x)e^{-2\pi ixy}\int_{-\infty}^{\infty} b(x)e^{-2\pi ixy}\,dx\,dt \\
&= \mathfrak{F}(a(x))\mathfrak{F}(b(x)). \qquad (A.2)
\end{aligned}$$

□

Lemma A.2. *(Also Lemma 2.2) The scaling property of the Fourier transform with a constant c is:*

$$\mathfrak{F}(\phi(cx)) = \frac{1}{|c|}\cdot\widehat{\phi}\left(\frac{y}{c}\right) \qquad (A.3)$$

Proof.

$$\begin{aligned}
\mathfrak{F}(\phi(cx)) &= \int_{-\infty}^{\infty}\phi(cx)e^{-2\pi ixy}\,dx \\
&= \frac{1}{c}\int_{-c\infty}^{c\infty}\phi(u)e^{-2\pi iy\frac{u}{c}}\,du. \qquad (A.4)
\end{aligned}$$

If $c > 0$ then

$$\frac{1}{c}\int_{-\infty}^{\infty}\phi(u)e^{-2\pi i\frac{y}{c}u}\,du = \frac{1}{c}\mathfrak{F}\left(\frac{y}{c}\right) = \frac{1}{|c|}\mathfrak{F}\left(\frac{y}{c}\right), \qquad (A.5)$$

while if $c < 0$ we have

$$\frac{1}{c}\int_{\infty}^{-\infty}\phi(u)e^{-2\pi i\frac{y}{c}u}\,du = -\frac{1}{c}\int_{-\infty}^{\infty}\phi(u)e^{-2\pi i\frac{y}{c}u}\,du = \frac{1}{-c}\mathfrak{F}\left(\frac{y}{c}\right) = \frac{1}{|c|}\mathfrak{F}\left(\frac{y}{c}\right). \qquad (A.6)$$

□

Lemma A.3. *(Also Lemma 2.5) The second derivative of the convolution between $a(x)$ and $b(x)$ is*

$$(a*b)''(x) = (a*b'')(x). \qquad (A.7)$$

Proof. As our functions are nice, we can interchange the derivatives and the integration, and find

$$\begin{aligned}
(a * b)''(x) &= \frac{d^2}{dx^2} \int_{-\infty}^{\infty} a(t) b(x-t)\, dt \\
&= \int_{-\infty}^{\infty} \frac{d^2}{dx^2} a(t) b(x-t)\, dt \\
&= \int_{-\infty}^{\infty} a(t) \frac{d^2}{dx^2} b(x-t)\, dt \\
&= \int_{-\infty}^{\infty} a(t) b''(x-t)\, dt \\
&= (a * b'')(x). \qquad \text{(A.8)}
\end{aligned}$$

□

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