S.T. Yau High School Science Award
Research Report

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Title of Research Report
On an extension of the Square Peg Problem for polygonal curves in higher dimensions

Date
August 20th, 2023

ON AN EXTENSION OF THE SQUARE PEG PROBLEM FOR POLYGONAL CURVES IN HIGHER DIMENSIONS

DERUI CHU

ABSTRACT. We investigate an extension to the unsolved Square Peg Problem, first proposed by Otto Toeplitz [14]. We study the problem for polygonal curves in higher dimensions, and prove that there exists an inscribed square-like tetrahedron in every polygonal curve in three-dimensional space.

KEYWORDS. Square Peg, inscribed square, topology, polygonal curves, linking number, transversality

Date: August 2023.

Acknowledgements

I would like to acknowledge and thank my supervising teacher, Dr. Bassirou Diatta, for providing me with valuable feedback and support throughout the research process.

Declaration of Academic Integrity

The participating team declares that the paper submitted is comprised of original research and results obtained under the guidance of the instructor. To the team's best knowledge, the paper does not contain research results, published or not, from a person who is not a team member, except for the content listed in the references and the acknowledgment. If there is any misinformation, we are willing to take all the related responsibilities.

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Date: August 19th, 2023

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We hereby declare that we

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3. observe the common standard of academic integrity adopted by most journals and degree theses.
4. have declared all the assistance and contribution we have received from any personnel, agency, institution, etc. for the research work.
5. undertake to avoid getting in touch with assessment panel members in a way that may lead to direct or indirect conflict of interest.
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Contents

1. Introduction

The Square Peg Problem, or inscribed square problem, asks whether for any planar curve γ there exists an inscribed square in γ. Here, inscribed means that the four vertices of the square must all lie in γ, but the square is not necessarily contained in the region bounded by γ.

The problem was first proposed by Otto Toeplitz [14] in 1911, and in spite of its long history, it remains open in full generality. So far, progress has only been possible by assuming some additional condition on the curve γ. In the case where γ is convex, an early partial result was obtained by Emch [3], who proved in 1916 that Toeplitz's conjecture holds for piecewise analytic curves with finite singularities. The case for convex curves was later completed by Christensen [2] and Zindler [15]. Numerous other accomplished mathematicians have attempted the problem and achieved partial solutions. These various results were proven through the use of techniques from a variety of mathematical fields, and have sought a range of different constraints and regularity conditions for their theorems.

In 1944, Shnirelman [11] proposed a version of the problem restricted to curves with a continuity condition on their curvature, to which he provided a flawed and incomplete proof that Guggenheimer [4] later revised in 1965, still relying heavily on the bordism argument that Shnirelman first used, as well as parts of trigonometry and topology. However, Guggenheimer's proof of Shnirelman's proposition was not all-encompassing, as it failed to address flaws in the limiting argument that appeared upon deformation of the curve.

During the second half of the 20th century and onwards, many more mathematicians proved the theorem under various regularity conditions. Jerrard's [6] paper from 1961, proved the theorem for the class of periodic analytic curves in the plane. Using the periodic nature of these functions, and constructing a function with each vertex of the curve, he analyzed the number of defined values and the supports at various points of such functions. After completing the initial proof for the existence of an inscribed square, he expanded his conjecture by proving that there exists precisely an odd number of squares, in large part by analyzing the parity of so-called "ordinary" values, defined by Jerrard as the number of vertices such that a 90-degree rotation around that vertex produces a curve that crosses the boundary of the original curve.

In 1989, Stromquist [12] provided a proof for smooth curves in R^n, which he extended to all locally monotone curves in the latter half of his paper.

In 2017, Tao [13] published a computationally intensive paper with a significant breakthrough, proving the theorem for all curves that are the union of two Lipschitz graphs — among the least regularity yet out of all correct proofs on Toeplitz's proposition to date. His proof uses complex signed integrals that benefited the use of his limiting argument.

Matschke [7] [8] was another major contributor to the progress in this problem. In two separate papers, he proved the problem for, among other classes, curves that do not contain an even number of sufficiently small trapezoids.

Broadly speaking, the results above all use tools from algebraic topology. During the problem's 112-year history, there have been many other attempts towards solutions with less regularity, or even complete proofs of Toeplitz's proposition. Many of these have fallen short due to a heavy reliance on a limiting argument that proved to be flawed, as the squares would become degenerate as opposed to converging to inscribe the various classes of curves as these conjectures would have expected. Nonetheless, Pak's paper [9], published in 2008, is a fairly recent example of a successful application of ideas from algebraic topology and the limiting argument. His proof that all simple, closed polygons in the plane inscribe a square involves simple techniques and concepts from geometry and set theory, which he applied in a very sophisticated manner. Pak took two points on a curve and constructed an isosceles right triangle with the chord as a leg, which gave him two choices for the position of the triangle's third vertex. These two choices correspond to two sets which represented the pairs that would admit an inscribed triangle. By considering the configuration space of polygonal curves and parameterizing this space by a torus (see Section 2.4 for further explanation), Pak was able to use the algebraic topology of the torus to show that the two sets must intersect, thereby proving the existence of an inscribed square.

A more recent publication was completed by Cantarella et al. [1] in 2021. They investigated a possible generalization of the Square Peg Problem to curves in higher dimensions, and asked whether there exists a so-called "square-like quadrilateral" inscribed in all closed, continuous curves in R^n. Motivated by their work as well as Pak's, we investigate an extension to the Square Peg Problem for polygonal curves in three dimensions, and ask whether such curves must admit a square-like tetrahedron, defined as a tetrahedron with congruent isosceles right triangles as two of its faces, sharing an edge at a leg of each triangle, with the right angles on opposite sides of the shared edge, as shown in Figure 1 below.

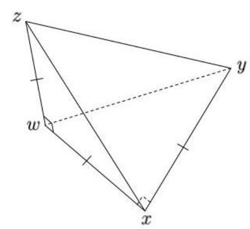

FIGURE 1. Square-like tetrahedron

This paper seeks to prove the following theorem:

Theorem 1. *Every simple polygonal curve in three-dimensional space has an inscribed square-like tetrahedron.*

2. PRELIMINARIES

2.1. **Setup.** Let P be an oriented polygonal curve in $\mathbb{R}^3$ with n vertices, labelled $p_1, p_2, \ldots, p_n$. For an ordered pair (w, x) of points $w, x \in P$ denote by y and z the other two vertices of a square-like tetrahedron $[wxyz]$ in $\mathbb{R}^3$, where $|\overline{wz}| = |\overline{wx}| = |\overline{xy}|$, and $\measuredangle zwx = \measuredangle yxw = 90$. Parameterize P by the circle and consider (w, x) as a point on a torus $T = P \times P$. Denote by $Y \subset T$ the subset of pairs (w, x) so that $y \in P$. Similarly, denote by $Z \subset T$ the subset of pairs (w, x) so that $z \in P$.

2.2. **Linking number.** A significant section in this paper relies on the concept of a *linking number*. Suppose γ_1, γ_2 are two non-intersecting simple closed curves in $\mathbb{R}^3$. If $r_1, r_2 : [0, 2\pi] \longrightarrow \mathbb{R}^3$ are parametrizations of γ_1, γ_2, respectively, we define their linking number as:

$$\text{link}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\det(r_2 - r_1, r_2', r_1')}{|r_1 - r_2|^3} \, dt_1 dt_2. \tag{1}$$

Although it is not clear at first, the above integral is always an integer. The linking number is so called because it corresponds to the number of times that two closed curves wind around each other. Depending on orientation, the linking number can be both positive and negative, though our proof will only require differentiation between the cases where two curves do and do not wrap around each other — having a non-zero and zero linking number, respectively. *e.g.* [10]

2.3. **The sense of 'generic'.** For some fixed integer n, one may consider the space of n-polygonal curves and equip it with a metric topology as follows.

First, we define a distance between two n-polygonal curves as follows: let K and Q be two polygonal curves with n vertices each, labelled as $K_1, K_2, \cdots, K_n$, and $Q_1, Q_2, \cdots, Q_n$, respectively. Define the distance $d(K, Q)$ between K and Q as

$$d(K,Q) = \min_{\pi} \sum_{i=1}^{n} |K_i - Q_{\pi(i)}| \tag{2}$$

where minimum is taken among all permutations $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$. In this distance function, a sequence of n-polygonal curves $P_n = \{P_1, P_2, P_3, \cdots\}$ converges to a n-polygonal curve P if and only if

$$\lim_{k\to\infty} d(P, P_k) = 0 \tag{3}$$

for some pairing of vertices between P and P_n.

One checks straightforwardly that d is a metric and from it, one can define a topology on the space of n-polygonal curves. A generic set of polygons is a set that is open and dense in this topology. In this paper, we address a generic case first, followed by the general case.

2.4. **Parameterization of the torus.** The torus is parameterized by length with reference to P and is treated as a square for the entirety of the proof, where every point on T corresponds to the ordered pair (y, z). This is visualized in Figure 2 below.

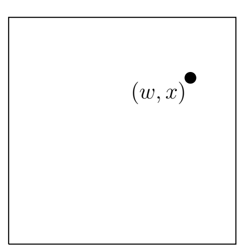

FIGURE 2. $T = P \times P$

3. Proof of the generic case

Theorem 2. *Every generic simple polygonal curve in three-dimensional space has an inscribed square-like tetrahedron.*

Having established the definitions of the sets Y and Z, our goal now is to show that they intersect.

First, we see that for a generic P, the set of points $Y_x = \{w : (w, x) \in Y\}$ is finite. This can be shown through a sphere argument involving two lemmas.

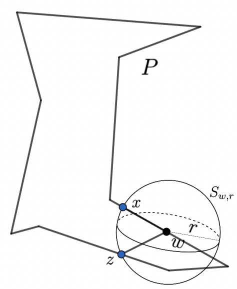

Figure 3. $S_{w,r}$

Let $S_{w,r}$ be a sphere centered at w with radius r, where $r > 0$. Such a sphere is shown above in Figure 3. We have the following definition:

Definition 3.1 (Qualifying radius)**.** A radius r is categorized as a *qualifying radius* if there are points $h_1, h_2 \in S_{w,r} \cap P$ that are equidistant from w, and such that $h_1 w$ is perpendicular to $h_2 w$. The pair (h_1, h_2) *qualifies* r.

Lemma 3. *For all $w \in P$, there exist finitely many qualifying radii r.*

Proof. Let A be the set of radii r that satisfy the condition described in the lemma. By contradiction, assume A is an infinite set. Since for some $R > 0$ sufficiently large, $S_{w,r} \cap P = \varnothing$ for all $r > R$, we have that A is bounded. Thus, by the Bolzano-Weierstrass Theorem, given an infinite number of satisfying values in the bounded set A, there must be an accumulation point in the interval $[0, R]$. This entails the existence of some converging sequence of r values that fit the aforementioned condition. However, this corresponds to a non-generic case, since it means that for some two radii r_1 and r_2 ϵ-close to each other where the intersections with P lie on the same edges of P, these intersections form a 90-degree angle at w for both radii. This can only be achieved if some two sides of P are perpendicular to each other. ☐

Lemma 4. *For every qualifying radius r, there exists only a finite number of pairs (h_1, h_2) that qualify r.*

Proof. Let r_q be a qualifying radius. Since S_{w,r_q} is a two-dimensional object while P is one-dimensional, we know that generically, by the transversality condition [5], they intersect at finitely many points. $\square$

Combining the claims in Lemma 3 and Lemma 4, we can now conclude that the set $Y_w = \{x : (w, x) \in Y\}$ is finite for every generic P that does not have orthogonal edges.

Lemma 5. *Y is composed of closed curves that are piecewise C^1-smooth.*

Proof. We argue by continuity. Start with some $w_g, x_g, y_g \in P$, where the three points form a right isosceles triangle with the right angle at x_g. Construct a sphere $S_{x,r}$ centered at x with radius r, where $x = x_g$ and $r = |\overline{w_g x_g}|$. Clearly, this sphere passes through w_g and y_g. In the generic case, as we locally increase and decrease r around $r = |\overline{w_g x_g}|$, the angle should be monotonically increasing or decreasing locally around $r = |\overline{w_g x_g}|$. Thus, by continuity, as w_g shifts along P by a function parameterized by length, there should also exist a radius very close to x_g such that the angle at x is 90 degrees.

$\square$

Having shown that Y is a union of piecewise differentiable closed curves, we will abuse notation slightly and refer to Y as a union of closed *polygonal arcs*.

Lemma 6. *The closed polygonal arcs in Y are simple and disjoint.*

Proof. We prove this lemma by contradiction, by assuming that some two closed polygonal arcs in Y do intersect or that some polygonal arc self-intersects. There exists four different cases for the relationship between any two closed curves in Y that have such an appearance. We address them one by one to show that none of the cases are both generic and achievable.

First, we examine the case where two polygonal arcs in the torus T intersect edge to edge. This cannot happen, because it would mean that the intersections w and x change directions along an edge of P, which is impossible since any point on Y can only move from one arc to another when w or x pass through a vertex of P.

The next case is where two polygonal arcs in T intersect vertex to edge. This case cannot happen by similar logic, since the intersections w and x cannot change directions unless at least one of them passes through a vertex on P.

The third case is where two polygonal arcs in T intersect vertex to vertex. This case is non-generic, since it can only be achieved when there exists some r and w such that $S_{w,r}$ passes through two vertices of P, where the vertices form a 90-degree angle at w.

Therefore, the only case remaining is where no two curves in Y intersect each other, and no curve self-intersects. In this case, w and x cannot disappear unless $S_{w,r}$ passes through a vertex of P, which is what forms the boundary of the closed curves in Y. □

From here, we also observe that Y does not pass through the diagonal in the torus T, since otherwise it would imply a non-generic case where at some vertex of P, the triangles all converge to the same point.

Lemma 7. *At least one of the closed polygonal arcs in Y is not null-homotopic.*

Proof. Choose some $x \in P$. Draw $S_{x,r}$, where r is infinitesimally small, and two of the sphere's intersections with P be w and y. For any x that is not a vertex on P, $\measuredangle wxy$ is 180 degrees with the radius infinitesimally small. As r increases, there exists a least R such that for all $S_{x,r}$ such that $r > R$, there are no intersections between the sphere and P. For $S_{x,R}$, $\measuredangle wxy$ is 0 degrees. Thus, there must be an odd number of radii for which $\measuredangle wxy$ measures 90 degrees. Thus, for a generic x, Y_x contains an odd number of elements. Therefore, at least one of the closed polygonal arcs in Y must not be null-homotopic. □

As can be checked with the same arguments, the lemmas proven above for Y also apply to Z. In summary, Y and Z are both a disjoint union of simple closed polygonal arcs, and at least one curve in Y as well as in Z are not null-homotopic. We denote one such curve in each of Y and Z as Y' and Z', respectively. Since Y' and Z' are also simple and do not intersect the diagonal Δ, we conclude that they must be homotopic to Δ.

Definition 3.2 (Linked; unlinked)**.** Two curves are *linked* if their linking number is nonzero. Else, we say they are *unlinked*.

Lemma 8. *Y' intersects Z'.*

Proof. We perform this proof by contradiction: assume that Y' does not intersect Z'. Let C_w be the circle centered at w with radius $|\overline{wx}|$, perpendicular to $\overline{wx}$, and let C_x be the circle centered at x with radius $|\overline{wx}|$, perpendicular to $\overline{wx}$. It is easy to see that for the pairs $(w, x) \in T$ that lie close to Δ, the linking number of C_w and P, as well as of C_x and P, are both nonzero. Keeping w fixed and moving x further away, there exists some distance β between the two points, such that C_x and P become unlinked for every $|\overline{wx}| > \beta$, with a linking number of 0. The same applies when keeping x fixed, moving w, and examining the linking number between C_w and P. This means that as the distance between w and x increases continuously — meaning the radii of C_x and C_w are also increasing identically, by setup —, the linking number must have been 1 or -1 for some radii. This is precisely the case where C_x or C_w are tangent to P, allowing the formation of right isosceles triangles wxy and xwz. We can also observe that when we keep w fixed and move x further from w, C_w will always become unlinked with P before C_x becomes unlinked with P. We can now divide the T into three regions:

- T_l: linked. C_w and C_x are both linked to P.
- T_s: semi-linked. C_w is unlinked and C_x is linked.
- T_u: unlinked. C_w and C_x are both unlinked to P.

This is shown in Figure 4 below.

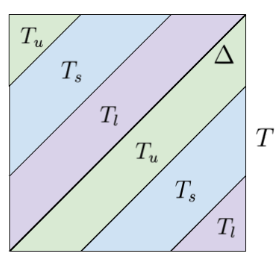

FIGURE 4. Three regions in T

Now, consider the smallest isosceles right triangle M inscribed in P. We start with w and x very close to each other, and move x further away while keeping w fixed. M can be labelled in two ways:

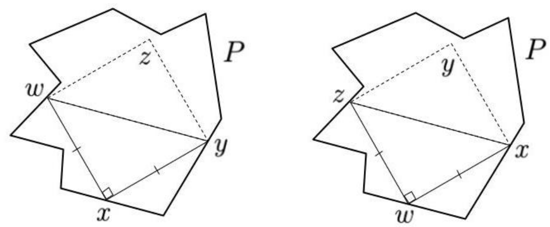

FIGURE 5. Two labellings for M

In the first labeling, y first appears on P when x and y are both vertices of M. From our previous findings, this means that C_w and P are unlinked. In the second labeling, z first appears on P when x and z are vertices of M. From our previous findings, this means that C_x are P are linked. According to the hypothesis, these observations imply that the two labelings should represent points on two different boundaries of the torus — namely, the first labelling should correspond to a point between T_s and T_u, while the second labeling should correspond to a point between T_l and T_s. However, our diagram in Figure 5 shows that the two labelings should both be in the boundary between T_l and T_s, since M can correspond to an element in both Y and Z. Therefore, we have reached a contradiction.

□

4. Finishing the proof of Theorem 1: the non-generic case

We complete the proof of Theorem 1 by addressing the case where P is non-generic. Because our conjecture restricts P to be a simple closed polygon, we are able to execute the following argument. For some non-generic polygonal curve P, there exists a sequence of generic polygonal curves P_n that converges to P in the distance sense, as defined in Section 2.3. By Theorem 2, for each n, there are square-like tetrahedrons Q_n inscribed on P_n. Our proof will be done once we show that Q_n does not degenerate as $P_n \to P$ and thus must converge to a square-like tetrahedron Q that is inscribed in P.

Lemma 9. *The square-like tetrahedrons Q_n do not degenerate as $P_n \to P$.*

Proof. By contradiction, suppose Q_n is degenerating. In this case, we can find right isosceles triangles T_n in Q_n which are degenerating to a point.

Since $P_n \longrightarrow P$, the vertices of P_n stay at a bounded distance from each other as $n \to \infty$. And because P_n is generic, the three vertices of the triangles

T_n all must lie on different sides of P_n. But this is a contradiction since T_n degenerating to a point would imply that vertices or sides of P_n would have to come together as $n \longrightarrow \infty$. $\square$

Our proof of Theorem 1 is now complete.

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