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Title: Finite orbits of the braid group actions

Finite orbits of the braid group actions

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Abstract

In this paper, we study the finite orbits of the braid group B_n action on the space of $n \times n$ upper-triangular matrices with 1's along the diagonal. On one hand, we give a necessary condition for a matrix M to be in a finite orbit; on the other hand, we classify and provide lengths of finite orbits in low-dimensional matrices and some other important cases.

Keywords: Braid group action, Finite orbits on upper-triangular matrices, Chebyshev polynomials

Table of Contents

1 Introduction

The braid group B_n is generated by $n-1$ generators $\sigma_1 \dots \sigma_{n-1}$ with the relations

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \;\;,\;\; \text{if } |i-j|>1. \end{aligned} \tag{1}$$

Following [3], the braid group B_n acts on the space U_+ of $n \times n$ upper-triangular matrices with 1's on the diagonal in the following way: for any $M \in U_+$, the entries of the matrix $\sigma_i(M)$, obtained by the action of the generator σ_i on M, are given by

$$\begin{aligned} \sigma_i(M)_{i,i+1} &= -M_{i,i+1} \\ \sigma_i(M)_{ij} &= -M_{i,i+1}M_{ij} + M_{i+1,j}, \;\; i+1<j \\ \sigma_i(M)_{ji} &= -M_{i,i+1}M_{ji} + M_{j,i+1}, \;\; j+1<i \\ \sigma_i(M)_{i+1,j} &= M_{ij}, \;\; i+1<j \\ \sigma_i(M)_{j,i+1} &= M_{ji}, \;\; j+1<i \\ \sigma_i(M)_{jk} &= M_{jk}, \;\; j<k, \;\; \text{and } j,k \neq i,i+1. \end{aligned} \tag{2}$$

The braid group acts from right to left on matrices: $\beta_2\beta_1(M) = \beta_2(\beta_1(M))$, $\beta_1, \beta_2 \in B_n$.

An important problem is to classify the finite orbit of the braid group action on U_+. The problem was motivated by the study of the geometry of 2D topological field theory [3, Appendix F], and the study of algebraic solutions of some nonlinear differential equations, see for example, [2, 4]. Past results in finding finite orbits have been focused on either the low dimension of 3×3 matrices [4], or specific types of matrices [7]. Rather generalized classifications of finite orbits in [5, 6, 8] focus on more abstract properties.

In this paper, we study the finite orbits in full generality. On one hand, we give a necessary condition on a matrix $M \in U_+$ such that it is in a finite orbit; on the other hand, we classify finite orbits in low-dimensional cases and in other important cases, and provide lengths and representatives for each orbit.

In this paper, we refer to the set of all $n \times n$ matrices that generate finite orbits of B_n as $FinOrb(B_n) \subseteq U_+$ and denote $Orb(M)$ as the finite orbit in U_+ generated by $M \in FinOrb(B_n)$ under the braid group.

1.1 A characterization of the matrices $M \in FinOrb(B_n)$

Our main result in this paper is

Theorem 1.1. *If a matrix $M \in FinOrb(B_n)$, and there exists an element M' in the finite orbit $Orb(M)$ such that $\sigma_i^2(M') \neq M'$ for some $i \in \{1, ..., n-1\}$, then all entries of M are in the form of $2\cos(r\pi)$ with $r \in \mathbb{Q}$.*

This theorem generalizes [7, Theorem 1 and Theorem 2], which prove that all matrices $M \in FinOrb(B_n)$, with the extra constraint $rank(M + M^T) = 2$ (i.e. all 3×3 submatrices are degenerate), mush have the form as in Theorem 1.1.

Note that the condition in Theorem 1.1 excludes the case $\sigma_i^2(M') = M'$ for all $i = 1, ..., n-1$ and $M' \in Orb(M)$. In this case, the braid group B_n action on the finite orbit $Orb(M)$ becomes the symmetric group S_n action. A complement to the Theorem 1.1 is as follows. We denote the set of matrices $M \in U_+$ such that $\sigma_i^2(\beta(M)) = \beta(M)$ for all $\beta \in B_n$ and $i = 1, ..., n-1$ (i.e., M generates a finite S_n orbit) as $FinOrb(S_n)$. We give a complete description of the set $FinOrb(S_n)$ in Section 3 by proving

Theorem 1.2. *If $M \in U_+$ has only $k \leq \lfloor \frac{n}{2} \rfloor$ non-zero entries*

$$|M_{12}| = a_1,\ |M_{34}| = a_2,\ \ldots\ldots,\ |M_{2k-1,2k}| = a_k \in \mathbb{R}, \tag{3}$$

then $M \in FinOrb(S_n)$. And the orbit's length, $|Orb(M)|$, is

$$\frac{n!}{(n-2k)! \prod_{i=1}^{s} r_i!}$$

where s is the number of the different values of $a_1, \ldots, a_k$, and $r_1, \ldots r_s$ denotes the cardinality of the s different values. Conversely, if $M \in FinOrb(S_n)$, then there exists $\beta \in B_n$, an integer $k \leq \lfloor \frac{n}{2} \rfloor$ and nonzero numbers $a_1, ..., a_k$ such that $\beta(M)$ has only non-zero entries as in (3).

1.2 Application of Theorem 1.1: new finite orbits and their classification

Theorem 1.1 gives a necessary condition for matrices $M \in FinOrb(B_n)$ that dramatically reduces the difficulty of seeking for finite orbits. In the second part of the paper, we use Theorem 1.1 to study and classify some special finite orbits. As far as we know, the finite orbits we obtain are new.

The case $r = 0, 1/2, 1$ in Theorem 1.1. The first case concerns of finite orbits on matrices with the entries ± 2 and 0, aside from 1's on the diagonal. We provide a classification of them and a formula that calculates the length for each orbit.

Theorem 1.3. *(1) The total number of the finite orbits of B_n on U_+, that only contain matrices with upper-triangular entries 0 or ± 2, is*

$$1 + \sum_{m=2}^{n} \sum_{d=1}^{\lfloor \frac{m}{2} \rfloor} p_d(m-d).$$

Here, $p_d(m-d)$ stands for the number of partitions of $m-d$ into d integers.

(2) Each of such finite orbits corresponds to a partition of $m-d$ into d integers. The length of the orbit, corresponding to a partition consisting of r_1 copies of t_1's, r_2 copies of t_2's, ..., and r_s of t_s's with $t_1, \ldots t_s$ being distinct integers, is

$$\frac{n! \cdot \prod_{i=1}^{s} 2^{t_i r_i}}{(n-m)! \cdot \prod_{i=1}^{s} (r_i!\,(t_i+1)!)^{r_i})}. \tag{4}$$

The case $r = \pm 1/3, 1/2$ in Theorem 1.1. The second case concerns of finite orbits on matrices with entries ± 1 and 0. Seemingly similar to the first case, there is a much greater variety of orbits. We only claim one conjecture here, and want to leave the study of it to a future work. The conjecture has been verified up to 9×9 case using the computer search algorithm described in section 4.4.

Conjecture 1.4. *The $n \times n$ Jordan block J_n (1's on the diagonal and sub-diagonal) generates a finite orbit (under the action of B_n) of length*

$$|Orb(J_n)| = 2^{n-1} \cdot (n+1)^{n-2}.$$

Moreover, any matrix in a Jordan form (1's and 0's on the sub-diagonal) generates a finite orbit.

Beyond matrices of Jordan form, we found all finite orbits in $4 \times 4, 5 \times 5, 6 \times 6, 7 \times 7$ matrices with entries ± 1 and 0 using a computer search algorithm that we developed. See Table 1 - 4 at the end of this paper.

Certain finite orbits in 5×5 matrices. We also classify all finite orbits generated by 5×5 matrix $M \in T(B_5)$ whose three 3×3 submatrices

$$M_1 = \begin{bmatrix} 1 & M_{12} & M_{13} \\ 0 & 1 & M_{23} \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & M_{23} & M_{24} \\ 0 & 1 & M_{34} \\ 0 & 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & M_{34} & M_{34} \\ 0 & 1 & M_{45} \\ 0 & 0 & 1 \end{bmatrix}$$

satisfy the determinant $|M_1 + M_1^T|, |M_2 + M_2^T|, |M_3 + M_3^T| \neq 0$ and $M_1, M_2, M_3 \in T(B_3)$.

The classification is done, again, by a computer search algorithm.

All the orbits found by our own computer program are listed at the end of the paper.

The finite orbits in 4×4. In the end, with the help of Theorem 1.1, we proved

Theorem 1.5. *If $M \in FinOrb(B_4), M \notin FinOrb(S_4)$ and $Orb(M)$ is non-degenerate, then any entry of the matrix is in the set*

$$\{2\cos(\frac{p\pi}{q}) \mid q = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 20, 25; p \in \mathbb{Z}\}.$$

This result makes it possible for a computer algorithm to search for all 4×4 finite orbits.

2 A characterization of the matrices that generate finite B_n orbits

The first observation is that, for a matrix $M \in U_+$ to be in a finite orbit of the braid group action, there must be

$$\sigma_i^k(M) = M$$

for any $i \in \{1, \dots n-1\}$ and an integer k depending on the chosen i. By formula (2), the generator σ_i^k only acts on two rows and two columns of the matrix M in individual triples $(M_{ii+1}, M_{ij}, M_{i+1j})$ and $(M_{ii+1}, M_{ji}, M_{ji+1})$ where $i < j$ and $i, j \in \{1, \dots n-1\}$. The action on these two triples are symmetrical, so we only study σ_i^k acting on $(M_{ii+1}, M_{ij}, M_{i+1j})$ for certain fixed $i, j \in \{1, \dots n-1\}$.

2.1 Odd powers: $\sigma_i^{2t-1}(M) = M$

Now assume that $\sigma_i^{2t-1}(M) = M$ for some positive integer t. On the one hand, by the assumption we have

$$\sigma_i^{2t-1}(M)_{i,i+1} = M_{i,i+1}, \tag{5}$$

while on the other hand, by formula (2), $\sigma_i(M)_{i,i+1} = -\sigma(M)_{i,i+1}$ that gives

$$\sigma_i^{2t-1}(M)_{i,i+1} = -\sigma_i^{2t-2}(M)_{i,i+1} = \cdots = \sigma_i(M)_{i,i+1} = -(M)_{i,i+1}. \tag{6}$$

The identities (5) and (6) then imply $M_{ii+1} = 0$. Now with $M_{ii+1} = 0$, we have $\sigma_i^{2t-1}(M)_{ij} = \sigma_i(M)_{ij} = M_{i+1j}$ and $\sigma_i^{2t-1}(M)_{i+1j} = \sigma_i(M)_{i+1j} = M_{ij}$ for $j = i+1, \dots n$.

Evidently, $\sigma^{2t-1}(M) = M$ yields the same conditions on M given any $t \in \mathbb{Z}^+$. This is the case of $M_{ii+1} = 2\cos(\frac{\pi}{2})$ in Theorem 1.1.

2.2 Chebyshev Polynomials

Now we analyze in more details the behaviors of the triple $M_{ii+1}, M_{ij}, M_{i+1j}$ $(i, j \in \{1, \dots n-1\}, i < j)$ after applying powers of σ_i to a M. It follows from the formula (2) that the ij-entry of the matrix $\sigma_i^k(M)$ is a polynomial of the three variables $M_{ii+1}, M_{ij}, M_{i+1j}$. It motivates us to introduce

Definition 2.1. *Given any fixed $j \in \{i+1, \dots n\}$, let P_k denote the degree t polynomial in the three variables $M_{ii+1}, M_{ij}, M_{i+1j}$ given by*

$$P_k(M_{ii+1}) := \sigma_i^k(M)_{ij}. \tag{7}$$

For convenient notation, we write $M_{ij} = a$, $M_{i+1j} = b$, and $M_{ii+1} = x$ for the rest of this section, since we are only going to discuss solutions of polynomials with $M_{i+1j}, M_{ij}, M_{ii+1}$ regardless of the choices of i, j. Then when restricts to the entries

$M_{ij} = a$, $M_{i+1j} = b$, and $M_{ii+1} = x$, the action of σ_i on M becomes the transformation

$$\sigma_i : (x, a, b) \to (-x, -ax + b, a).$$

Lemma 2.2. *The polynomial*

$$Q_k(x) := \delta_k P_k(2x), \delta_k = \begin{cases} -1 : k \equiv 1, 2 \pmod 4 \\ 1 : k \equiv 0, 3 \pmod 4 \end{cases} \tag{8}$$

satisfies the induction formula

$$Q_{k+2} = 2x \cdot Q_{k+1} - Q_k \tag{9}$$

with the initial terms $Q_0(x) = a$ and $Q_1(x) = 2ax - b$.

The induction on $P_k(x)$, by the actions of the braid group (2), is

$$\begin{aligned} P_{2t+2} &= P_{2t+1} \cdot x + P_{2t}, \\ P_{2t+1} &= -P_{2t} \cdot x + P_{2t-1} \end{aligned}$$

with the first few terms being

$$P_0(x) = a, P_1(x) = -ax + b, P_2 = -ax^2 + bx - a, P_3 = ax^3 - bx^2 - 2ax + b, \ldots$$

Right now, $P_{2t}(x)$'s highest degree term alternates in signs as t increases. We adjust the highest-degree coefficient to be always positive and substitute x with $2x$ by defining $Q_k(x)$ as in (8). This unifies the original parity-dependent induction formula to the relation (9), with the first few terms now being

$$\begin{aligned} Q_0(x) &= a, \\ Q_1(x) &= 2ax - b, \\ Q_2(x) &= 4ax^2 - 2bx - a, \\ Q_3(x) &= 8ax^3 - 4bx^2 - 4ax + b, \\ Q_4(x) &= 16ax^4 - 8bx^3 - 12ax^2 + 4bx + a, \\ Q_5(x) &= 32ax^5 - 16bx^4 - 32ax^3 + 12bx^2 + 6ax - b. \end{aligned}$$

It finishes the proof. □

One can write down the explicit expression of $Q_k(x)$ as follows. Let us introduce the well-known Chebyshev polynomials of the second kind $U_k(x)$. It has the recursive formula

$$U_{k+2} = 2x \cdot U_{k+1} - U_k,$$

which is the same as the one for $Q_k(x)$ described in (9). $U_k(x)$ has the initial terms

$$U_0(x) = 1, U_1(x) = 2x.$$

The Chebyshev Polynomials of the second kind line up with the sequence of polynomials $Q_k(x)$ in the following way

$$Q_k(x) = a \cdot U_k(x) - b \cdot U_{k-1}(x).$$

It is known that the Chebyshev Polynomials of the second kind have the following explicit form (see [1])

$$U_k(x) = \frac{(x+\sqrt{x^2-1})^{k+1} - (x-\sqrt{x^2-1})^{k+1}}{2\sqrt{x^2-1}}.$$

Thus, we get

$$\begin{aligned} Q_k(x) &= a \cdot U_k(x) - b \cdot U_{k-1}(x) \\ &= \frac{(x+\sqrt{x^2-1})^k(ax+a\sqrt{x^2-1}-b) + (x-\sqrt{x^2-1})^k(-ax+a\sqrt{x^2-1}+b)}{2\sqrt{x^2-1}}. \end{aligned} \tag{10}$$

In summary, the polynomial $P_k(x)$ (or equivalently $Q_k(x)$) has a closed formula.

2.3 Even powers: $\sigma_i^{2t}(M) = M$

Suppose $\sigma_i^{2t}(M) = M$ for some $k \in \mathbb{Z}, i \in \{1, \dots n-1\}$). It imposes the following two equations for the entries M_{ij}, M_{i+1j}:

$$\begin{aligned} \sigma_i^{2t}(M)_{ij} &= M_{ij}, \\ \sigma_i^{2t-1}(M)_{ij} &= \sigma_i^{2t}(M)_{i+1j} = M_{i+1j}. \end{aligned} \tag{11}$$

Using our notations $M_{ij} = a$, $M_{i+1j} = b$, and $M_{ii+1} = x$, (11) is rewritten as

$$\begin{aligned} P_{2t}(x) &= a, \\ P_{2t-1}(x) &= b. \end{aligned} \tag{12}$$

In terms of Chebyshev polynomials of the second kind $U(x)$, (11) is rewritten as (recall the definition of δ in (8))

$$\begin{aligned} a \cdot U_{2t}(x) - b \cdot U_{2t-1}(x) &= \delta_{2t} a, \\ a \cdot U_{2t-1}(x) - b \cdot U_{2t-2}(x) &= \delta_{2t-1} b. \end{aligned} \tag{13}$$

The possible solutions x of the equation (13) characterize the possible subdiagonal element $(M)_{ii+1}$ in M.

Proposition 2.3. *(1) Assume that a and b are not zero at the same time, then the solution x of the equation* (13) *takes the form $x = 2\cos(r\pi)$ for some $r \in \mathbb{Q}$. To be more precise, the possible*

$$r = \begin{cases} \frac{p}{t}, \frac{2p+1}{4t} & \textit{for } p \in \mathbb{Z}, \quad \textit{if } t \in 2\mathbb{Z} \\ \frac{2p+1}{2t}, \frac{2p+1}{4t} & \textit{for } p \in \mathbb{Z}, \quad \textit{if } t \notin 2\mathbb{Z} \end{cases}, \tag{14}$$

(2) Conversely, for certain $i \in \{1, ..., n-1\}$, if

$$M_{ii+1} \in \{2\cos\left(\frac{p\pi}{t}\right); p \in \mathbb{Z}, t \in 2\mathbb{Z}\} \cup \{2\cos\left(\frac{(2p+1)\pi}{2t}\right); p \in \mathbb{Z}, t \notin 2\mathbb{Z}\},$$

then $\sigma_i^{2t}(M) = M$.

Remark 2.4. *The case $a = b = 0$, corresponding to the finite orbits of symmetric group S_n, will be thoroughly discussed in 3.*

Proof. First of all, we show that

Lemma 2.5. *All real solutions x are bounded by $|x| \leq 1$.*

Proof. If $x > 1$, we have

$$\lim_{k\to\infty}(x - \sqrt{x^2-1})^k = 0, \ \lim_{k\to\infty}(x + \sqrt{x^2-1})^k = \infty,$$

and furthermore by the expression (10), we have

$$\lim_{k\to\infty} Q_k(x) = \infty$$

as long as $ax + a\sqrt{x^2-1} - b \neq 0$. Similarly, if $x < -1$,

$$\lim_{k\to\infty} Q_k(x) = -\infty$$

as long as $-ax + a\sqrt{x^2-1} + b \neq 0$. Obviously, if $Q_k(x)$ approaches infinity, no finite orbit would exist. We are left with the edge cases to consider

$$\begin{aligned} ax + a\sqrt{x^2-1} - b = 0 \ (x > 1) \\ -ax + a\sqrt{x^2-1} + b = 0 \ (x < -1). \end{aligned} \tag{15}$$

Suppose $x > 1$, then the only way for (x, a, b) to be in a finite orbit is to satisfy the first equation in (15) , which gets simplified to

$$\sqrt{x^2-1} = \frac{b}{a} - x \ (a \neq 0)$$

squaring both sides,

$$x^2 - 1 = x^2 - \frac{2b}{a}x + \frac{b^2}{a^2} \ (a \neq 0)$$

which gives the following expression for x:

$$x = \frac{a^2+b^2}{2ab} \ (a, b \neq 0) \tag{16}$$

After the transformation $\sigma : (x, a, b) \to (-x, -ax + b, a)$, x is smaller than -1, so (x, a, b) also needs to satisfy the second equation in (15) after the transformation, which looks like

$$-(-ax + b)(-x) + (-ax + b)\sqrt{x^2 - 1} + a = 0$$

After simplification, we have

$$(-ax + b)\sqrt{x^2 - 1} = ax^2 - bx - a$$

Squaring both sides, we get

$$a^2x^4 - 2abx^3 + b^2x^2 - a^2x^2 + 2abx - b^2 = a^2x^4 - 2abx^3 + b^2x^2 + 2abx + a^2 - 2a^2x^2$$

which simplifies to

$$-a^2x^2 - b^2 = a^2 - 2a^2x^2$$

and gives an expression for x:

$$x = \pm\frac{\sqrt{a^2 + b^2}}{a} \ (a \neq 0) \tag{17}$$

We need both (17) and (16) to hold. By assumption, $a, b \neq 0$, we combine (17) and (16) to form

$$\pm\frac{\sqrt{a^2 + b^2}}{a} = \frac{a^2 + b^2}{2ab}$$

which reduces to

$$\pm\sqrt{3}b = a$$

Substituting this into (17), we have $x = \pm\frac{2}{\sqrt{3}}$, contradicting with our assumption $x > 1$. □

Now by Lemma 2.5, any solution to the system $P_{2t}(x) = a, P_{2t-1}(x) = b$ takes the form $x = \cos\phi$ for some real number $\phi \in \mathbb{R}$. From (13), ϕ satisfies

$$a \cdot U_{2t}(\cos\phi) - b \cdot U_{2t-1}(\cos\phi) = \delta_{2t}a$$
$$a \cdot U_{2t-1}(\cos\phi) - b \cdot U_{2t-2}(\cos\phi) = \delta_{2t-1}b$$

The Chebyshev Polynomials of the second kind satisfies the identity

$$U_k(\cos\theta)\sin\theta = \sin(k + 1)\theta.$$

So, we have

$$\begin{aligned} a\sin((2t + 1)\phi) - b\sin(2t\phi) &= \delta_{2t}a\sin\phi \\ a\sin(2t\phi) - b\sin((2t - 1)\phi) &= \delta_{2t-1}b\sin\phi \end{aligned} \tag{18}$$

The definition in (8) guarantees $\delta_{2t} = \delta_{2t-1}$, which we will just call δ from now on.

If $a = 0, b \neq 0$, we have

$$\begin{aligned} \sin(2t\phi) &= 0 \\ \sin((2t-1)\phi) &= 0 \end{aligned}$$

which has the only solutions $\phi = p\pi \ p \in \mathbb{Z}$.

If $a \neq 0, b = 0$, we have

$$\begin{aligned} \sin(2t\phi) &= 0 \\ \sin((2t+1)\phi) &= 0 \end{aligned}$$

which has the same solutions $\phi = p\pi \ p \in \mathbb{Z}$.

If $a, b \neq 0$, we manipulate (18) to obtain

$$\begin{aligned} a\left(\sin\left((2t+1)\phi\right) - \delta \sin\phi\right) &= b \sin(2t\phi) \\ a \sin(2t\phi) &= b\left(\sin\left((2t-1)\phi\right) + \delta \sin\phi\right) \end{aligned} \tag{19}$$

Suppose $\sin(2t\phi), (\sin\left((2t-1)\phi\right) + \sin\phi) \neq 0$, then moving a, b to the same side gives

$$\frac{b}{a} = \frac{\sin\left((2t+1)\phi\right) - \delta \sin\phi}{\sin(2t\phi)} = \frac{\sin(2t\phi)}{\sin\left((2t-1)\phi\right) + \delta \sin\phi} \tag{20}$$

eliminating the denominators,

$$\begin{aligned} &\sin((2t-1)\phi)\sin((2t+1)\phi) + \delta \sin\phi(\sin((2t+1)\phi) \\ &\qquad - \sin((2t-1)\phi)) - \sin^2\phi - \sin^2(2t\phi) = 0 \end{aligned} \tag{21}$$

Using the product-to-sum formula on $\sin((2t-1)\phi)\sin((2t+1)\phi)$ and $\sin^2(2t\phi)$, we simplify the left hand side of the equation to

$$\begin{aligned} &\sin((2t-1)\phi)\sin((2t+1)\phi) + \delta \sin\phi(\sin((2t+1)\phi) - \sin((2t-1)\phi)) - \sin^2\phi - \sin^2(2t\phi) \\ &= -\frac{1}{2}\cos(4t\phi) + \frac{1}{2}\cos(2\phi) + \delta \sin\phi(\sin((2t+1)\phi) - \sin((2t-1)\phi)) - \sin^2\phi + \frac{1}{2}\cos(4t\phi) - \frac{1}{2} \\ &= \frac{1}{2} - \sin^2\phi + \delta \sin\phi(\sin((2t+1)\phi) - \sin((2t-1)\phi)) - \sin^2\phi - \frac{1}{2} \\ &= \delta \sin\phi(\sin((2t+1)\phi) - \sin((2t-1)\phi)) \end{aligned}$$

We obtain equation $\sin((2t+1)\phi) = \sin((2t-1)\phi$ ($\sin\phi \neq 0$), which implies that $(2t+1)\phi = (2t-1)\phi + 2p\pi$ or $(2t+1)\phi = -(2t-1)\phi + (2p+1)\pi$ ($p \in \mathbb{Z}$). The first equation gives the same solution as earlier; the second equation gives the solution $\phi = \frac{2p+1}{4t}\pi$. We verify that this solution satisfies (18). Note that this solution depends on the values of a, b.

Now, suppose $\sin(2t\phi) = 0$. Since $a, b \neq 0$, we derive the following from (18):

$$\begin{aligned} \sin((2t+1)\phi) &= \delta \sin\phi \\ \sin((2t-1)\phi) &= -\delta \sin\phi \\ \sin(2t\phi) &= 0. \end{aligned} \tag{22}$$

The first equation is

$$\sin((2t+1)\phi) - \delta \sin\phi = 0$$

which can be rewritten with a substitution of the second equation $\sin((2t-1)\phi) = -\delta \sin\phi$:

$$\begin{aligned} \sin((2t+1)\phi) - \delta\sin\phi &= \sin 2\phi \cos((2t-1)\phi) + \sin((2t-1)\phi)\cos 2\phi - \delta \sin\phi \\ &= 2\sin\phi\cos\phi\cos((2t-1)\phi) - \delta\sin\phi(2\cos^2\phi - 1) - \delta\sin\phi \\ &= \sin\phi\cos\phi(\cos((2t-1)\phi) - \delta\cos\phi) \end{aligned}$$

Now, we have the equation

$$\cos((2t-1)\phi) = \delta\cos\phi \ (\sin\phi, \cos\phi \neq 0) \tag{23}$$

Combining (22) and (23), we have

$$\begin{aligned} \cos((2t-1)\phi) &= \delta\cos\phi \\ \sin((2t-1)\phi) &= -\delta\sin\phi \\ \sin(2t\phi) &= 0 \end{aligned} \tag{24}$$

If $\delta = 1$, then $t \in 2\mathbb{Z}$ and $(2t-1)\phi = -\phi + 2p\pi$, which gives the solution $\phi = \frac{p}{t}\pi$ $(p \in \mathbb{Z})$. If $\delta = -1$, then $t \notin 2\mathbb{Z}$ and $(2t-1)\phi = -\phi + (2p+1)\pi$, which gives the solution $\phi = \frac{2p+1}{2t}\pi$ $(p \in \mathbb{Z})$. It can be verified that these solutions do satisfy (18).

Doubling the value of common solutions of $Q(x)_{2t} = \delta a, Q(x)_{2t-1} = \delta b$, we get all the solutions to $P_{2t} = a, P_{2t-1} = b$, which are all in the form of $2\cos(\pi r), r \in \mathbb{Q}$ where the values of r are the ones described in Proposition 2.3.

Now we prove part (2) of the proposition. From the proof of part (1), the solutions such that $\sin(2t\phi) = 0$ satisfy $Q_{2t} = \delta a, Q_{2t-1} = \delta b$ for any real numbers a, b. They correspond to the following solutions of $P_{2t} = a, P_{2t-1} = b$:

$$\begin{cases} t \in 2\mathbb{Z} : \{2\cos(\frac{p\pi}{t}); p \in \mathbb{Z}\} \\ t \notin 2\mathbb{Z} : \{2\cos(\frac{(2p-1)\pi}{2t}); p \in \mathbb{Z}\} \end{cases}$$

setting the entry M_{ii+1} to a value above, we will always have $\sigma_i^{2t}(M) = M$, no matter the value of other entries.

□

2.4 Proof of Theorem 1.1

Note that Proposition 2.3 already implies Theorem 1.1 for any sub-diagonal entries. Since $\sigma_i\sigma_{i+1}\cdots\sigma_{j-1}(M)_{ij} = M_{ii+1}$ any $j > i$, all entries of M must be in the form of $2\cos(\pi r)$, where $r \in \mathbb{Q}$.

3 Classification of all finite S_n orbits

The generating relation of B_n, together with the relation $\sigma_i^2 = 1$ for $i = 1, \ldots n-1$, become the generating relation of S_n. By definition, a matrix $M \in FinOrb(S_n)$ if $\sigma_i^2(\beta(M)) = \beta(M)$ for all $i = 1, ..., n-1$ and $\beta \in B_n$. Thus, if $M \in FinOrb(S_n)$, then the orbit generated by M under the action of B_n is actually a finite S_n orbit. In this section, we classify such matrices and give a proof of Theorem 1.2.

Lemma 3.1. *Any matrix $M \in FinOrb(S_n)$ does not have any two non-zero entries on the same row or column.*

Proof. By formula (2), $\sigma_i^2(M) = M$ yields the following relation:

$$\begin{aligned} M_{ij} &= -M_{ii+1}^2 M_{ij} + M_{ii+1}M_{i+1j} + M_{ij} \\ M_{i+1j} &= -M_{ii+1}M_{ij} + M_{i+1j} \end{aligned} \tag{25}$$

where $j \in \{i+1, \ldots n\}$. The second equation gives $M_{ii+1}M_{ij} = 0$. Substituting this into the first equation, we obtain the solutions

$$M_{ii+1} = 0 \text{ or } M_{ij} = M_{i+1j} = M_{ji} = M_{ji+1} = 0 \;\; (i < j). \tag{26}$$

If M_{ii+1} is non-zero, all elements on the same row and column, all elements on the preceding column, and all elements on the succeeding row are zero by (26).

Furthermore, if both M_{ij}, M_{ik} $(i, j, k \in \{1, \ldots n-1\}, i < j < k)$ are nonzero, $M' = \sigma_{i+1}\cdots\sigma_{j-1}(M)$ has $M'_{ii+1}, M'_{ik} \neq 0$, which means that $M' \notin FinOrb(S_n)$. The same applies to two non-zero entries on the same column. It proves the lemma. ☐

The lemma motives the following definition.

Definition 3.2. *An $n\times n$ matrix $M \in U_+$ whose only non-zero entries are $M_{12}, M_{34}, \ldots M_{2k-1,2k}$, for some integer $k \leq \lfloor \frac{n}{2} \rfloor$, is called a standard matrix.*

Lemma 3.3. *Given any $M \in FinOrb(S_n)$, there exists a braid $\beta \in B_n$ and a standard matrix X such that $\beta(M) = X$. That is, any finite S_n orbit contains a standard matrix.*

Proof. Suppose $M \in FinOrb(S_n)$. Choose any non-zero entry M_{ij}, $i < j$. It is immediately true that $M_{i-1i} = 0$. Thus, $\sigma_{i-1}(M)_{i-1j} = M_{ij} \neq 0$. By the same logic, $\sigma_{i-1}(M)_{i-2,i-1} = 0$ and $\sigma_{i-2}\sigma_{i-1}(M) = M_{ij} \neq 0$. Repeating this process, we have $\sigma_1\cdots\sigma_{i-1}(M)_{1j} = M_{ij}$, and further more $\sigma_2\cdots\sigma_{j-1}\sigma_1\cdots\sigma_{i-1}(M)_{12} = M_{ij}$. No nonzero entry of $M' = \sigma_2\cdots\sigma_{j-1}\sigma_1\cdots\sigma_{i-1}(M)$ is in the first two rows because $M' \in FinOrb(S_n)$ and $M'_{12} \neq 0$.

We can apply the same method above and set $\sigma_4 \cdots \sigma_{l-1}\sigma_3 \cdots \sigma_{k-1}(M') = M''$, where $M'_{kl} \neq 0$ $(k, l \in \{1, \dots n\}, k < l)$. Since the generators σ_1, σ_2 are not used during this process, $M'_{12} = M''_{12} \neq 0$. We also have $M''_{34} \neq 0$. Hence, no nonzero entry of M'' is in the first four rows besides M''_{12}, M''_{34}.

The same process puts at most $\lfloor \frac{n}{2} \rfloor$ nonzero entries of M to the locations of nonzero entries in a standard matrix. If M has less than or equal to $\lfloor \frac{n}{2} \rfloor$ nonzero entries, then the method described above uses a certain braid $\beta \in B_n$ to achieve $\beta(M) = X$, where X is a standard matrix.

Now, suppose $M \in FinOrb(S_n)$ has $k > \lfloor \frac{n}{2} \rfloor$ nonzero entries. We may randomly choose $\lfloor \frac{n}{2} \rfloor$ of them and use the braid designed earlier to map them to entries on the subdiagonal. Specifically, we designed $\beta \in B_n$ such that $M^* = \beta(M)$ and $M^*_{12}, M^*_{34}, \dots \neq 0$. However, these $\lfloor \frac{n}{2} \rfloor$ nonzero entries don't allow any other entry to also be nonzero, since $M \in FinOrb(S_n)$. Therefore, M can have a maximum of $\lfloor \frac{n}{2} \rfloor$ nonzero entries by contradiction. □

A proof of Theorem 1.2:

We discuss the stabilizers of a standard matrix X.

We may "swap" two adjacent nonzero entries as such: $\sigma_i\sigma_{i-1}\sigma_{i+1}\sigma_i(X)_{i-1i} = X_{i+1,i+2}$ and $\sigma_i\sigma_{i-1}\sigma_{i+1}\sigma_i(X)_{i+1,i+2} = X_{i-1i}$. If $X_{i-1i} = X_{i+1,i+2}$, then $\sigma_i\sigma_{i-1}\sigma_{i+1}\sigma_i$ is a stabilizer of X; if $X_{i-1i} = -X_{i+1,i+2}$, then $\sigma_{i-1}\sigma_{i+1}\sigma_i\sigma_{i-1}\sigma_{i+1}\sigma_i$ is a stabilizer of X. Combinations of such braids permute all nonzero entries of X.

Let $a_i = |X_{2i-1,2i}|, i = 1, \dots k$. Denote the frequencies of all the distinct numbers in $a_1, \dots a_k$ as $r_1, \dots r_s$. Then, we have $r_i!$ stabilizers that permutes the r_i entries with the same absolute values for $i = 1, \dots s$.

Furthermore, a standard matrix X with $k < \lfloor \frac{n}{2} \rfloor$ non-zero entries only has zeros on the final $n - 2k$ columns. So, braid generators $\sigma_{2k+1}, \sigma_{2k+2}, \dots \sigma_{n-1}$ and their combinations are also stabilizers. There are $n - 2k - 1$ such stabilizing braids. Because these are generators of S_n, the $n - 2k - 1$ generators generate all permutations of $n - 2k$ elements, or all $(n - 2k)!$ stabilizers of X.

The two types of stabilizers do not intersect, so we find the total number of stabilizers to be

$$|Stab(X)| = (n - 2k)! \prod_{i=1}^{n} r_i!.$$

We apply the Orbit-Stabilizer Theorem on X, which generates an orbit of S_n by Lemma 3.3:

$$|Orb(X)| = \frac{|S_n|}{|Stab(X)|} = \frac{n!}{(n - 2k)! \prod_{i=1}^{n} r_i!}.$$

It finishes the proof of Theorem 1.2.

4 New finite orbits and their properties

Definition 4.1. *An orbit generated by $M \in FinOrb(B_n)$ such that $rank(M + M^T) > 2$ is called a degenerate orbit.*

Lemma 4.2. *If $M \in FinOrb(B_n)$, then for any $i, j, k \in 1, 2, \ldots n$ and $i < j < k$, the 3×3 submatrix*

$$M(i,j,k) := \begin{bmatrix} 1 & M_{ij} & M_{ik} \\ 0 & 1 & M_{jk} \\ 0 & 0 & 1 \end{bmatrix}$$

is in $FinOrb(B_3)$.

Proof. The braid $\beta_1 = \sigma_{k-2}\sigma_{k-3}\cdots\sigma_j$ applied on $M \in U_+$, $\beta_1(M) = M'$, results in $M'_{ik-1} = M_{ij}, M'_{ik} = M_{ik}, M'_{k-1k} = M_{jk}$. Then, applying the braid $\beta_2 = \sigma_{k-3}\sigma_{k-4}\cdots\sigma_i$ to get $M'' = \beta_2(M')$, we have $M''_{k-2,k-1} = M'_{k-2,k-1}, M''_{k-2,k} = M'_{ik}, M''_{k-1,k} = M'_{k-1,k}$. By formula (2), one sees that the 3×3 upper-triangular submatrix of M'' formed by three adjacent entries $M''_{k-2,k-1}, M''_{k-2,k}, M''_{k-1,k}$ generates a finite orbit. Thus, the original submatrix must also generate a finite orbit. □

This lemma will be particularly useful because the set $FinOrb(B_3)$ is well-researched: we have found all of $FinOrb(S_3)$; [7] finds all 3×3 degenerate orbits; [4] finds the remaining non-degenerate 3×3 orbits.

4.1 Proof of Theorem 1.3

There are only three 3×3 finite orbits of only entries 0 and ± 2:

$$T_1 := \{ \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \}, \tag{27}$$

$$T_2 := \{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \}, \tag{28}$$

and the third one being just $\{I\}$, an orbit of length 1. Note that $T_2, \{I\} \subseteq FinOrb(S_n)$.

4.1.1 Establish an isomorphism between matrices and partitions

Proposition 4.3. *Let $C \subseteq \{1, \ldots n\}$ be a subset, and let $Q(C, p)$ denote the set of all possible partitions of C into p subsets, where each subset has at least two elements. The subspace of $FinOrb(B_n)$ where all matrix entries are 0 or ± 2 besides the diagonal is* *isomorphic to*

$$\bigcup_{C \subseteq \{1,\dots n\}} \bigcup_{p=1}^{\lfloor \frac{|C|}{2} \rfloor} Q(C,p) \times \mathbb{Z}_{2^{(|C|-p)}}.$$

Proof. Let $G \in FinOrb(B_n)$ be a matrix such that all its entries are 0 or ± 2 besides the diagonal. We choose the first nonzero entry G_{x_j,y_j} of each row i, $i = 1, \dots n-1$, skipping the rows of all 0's. We end up with t entries $G_{x_1,y_1}, \dots G_{x_t,y_t}$ where $x_i < y_i$ for $i = 1, \dots, t$.

If there exists $y_i = y_j,\ i < j,\ i, j \in \{1, \dots n-1\}$, then by Lemma 4.2, we have $G_{x_i,x_j} = \pm 2$ to satisfy

$$\begin{bmatrix} 1 & G_{x_i,x_j} & G_{x_i,y_i} \\ 0 & 1 & G_{x_j,y_j} \\ 0 & 0 & 1 \end{bmatrix} \in T_1.$$

Since G_{x_i,x_j} is an entry before G_{x_i,y_i} on row i, we have contradiction. Hence, $y_1, \dots y_{n-1}$ are mutually distinct, and we may match pairs x_i, y_i with a bijective map $f : x_i \to y_i$ for $i = 1, \dots n-1$. With mutually distinct $x_1, \dots x_{n-1}$ and $y_1, \dots y_{n-1}$, we can always uniquely determine index i given x_i or y_i.

Now, we divide $C = \{x_1, \dots, x_t\}$ into subsets in the following algorithm:

Step 1: Create empty set C_{x_1};

Step 2: For the empty set C_{x_j} created, add x_j to it. Delete x_j from C;

Step 3: For the element x_k last added C_{x_j}, if there exists $x_l \in C$ such that $x_l = f(x_k)$, add x_l to C_{x_j}, delete x_l from C, and repeat Step 3. Otherwise, add $f(x_l)$ to C_{x_j} and proceed to Step 4;

Step 4: Create empty set C_{x_j}, where $x_j \in C$ has j being the smallest subscript among the remaining variables in C. Repeat Step 2, Step 3, and Step 4 until C is empty.

Now, for any subset $C_{x'_1} = \{x'_1 < \dots < x'_s < x'_{s+1}\} \subseteq C$ produced by our algorithm, note that $f^{i-1}(x'_1) = (x'_i), i = 1, \dots s+1$. Consider the matrix entries $G_{x'_1,f(x'_1)}, \dots G_{x'_{s+1},f(x'_{s+1})}$, which are all nonzero. Since $x'_{i+1} = f(x'_i)$, we know from Lemma 4.2 that

$$\begin{bmatrix} 1 & G_{x'_i,f(x'_i)} & G_{x'_i,f(x'_{i+1})} \\ 0 & 1 & G_{x'_{i+1},f(x'_{i+1})} \\ 0 & 0 & 1 \end{bmatrix} \in T_1,$$

for $i = 1, \dots s-1$. Once we know this and the fact that $G_{x'_i,y'_i}, G_{x'_{i+1},y'_{i+1}} \neq 0$, $G_{x'_i,f(x'_{i+1})} = G_{x'_i,f^2(x'_i)}$ is uniquely determined from 2 and -2.

Similarly, now for $i = 1, \dots s-2$,

$$\begin{bmatrix} 1 & G_{x'_i,f(x'_{i+1})} & G_{x'_i,f(x'_{i+2})} \\ 0 & 1 & G_{x'_{i+2},f(x'_{i+2})} \\ 0 & 0 & 1 \end{bmatrix} \in T_1$$

by Lemma 4.2. Thus, $G_{x'_i, f(x'_{i+2})} = G_{x'_i, f^3(x'_i)}$ is uniquely determined from 2, -2.

Using the same idea, we may uniquely determine the value of entries $G_{f^i(x'_1), f^j(x'_1)}$ for any $i, j \in \{0, \dots s\}, i < j$. We determine these entries for every subset obtained from C using our algorithm.

Now, suppose there exists $G_{x_0, y_0} \neq 0$ that is not uniquely determined by the method above. There definitely exists subset $C_{x_0} = \{x_0, f(x_0), \dots f^{s-1}(x_0), f^s(x_0)\} \subseteq C$, since row x_0 has at least one nonzero entry. If $f(x_0) = y_0$, we have contradiction. If otherwise, we know that

$$\begin{bmatrix} 1 & G_{x_0, f(x_0)} & G_{x_0, y_0} \\ 0 & 1 & G_{f(x_0), y_0} \\ 0 & 0 & 1 \end{bmatrix} \in T_1, G_{f(x_0), y_0} \neq 0 \tag{29}$$

by Lemma 4.2. If $y_0 < f^2(x_0)$, then $G_{f(x_0), f^2(x_0)}$ is not the first nonzero entry on row $f(x_0)$ and we have contradiction. If $y_0 = f^2(x_0)$, then $G_{x_0, y_0} = G_{x_0, f^2(x_0)}$ has been determined using the earlier method and we have contradiction. If $y_0 > f^2(x_0)$, then again,

$$\begin{bmatrix} 1 & G_{f(x_0), f^2(x_0)} & G_{f(x_0), y_0} \\ 0 & 1 & G_{f^2(x_0), y_0} \\ 0 & 0 & 1 \end{bmatrix} \in T_1, G_{f^2(x_0), y_0} \neq 0 \tag{30}$$

Eventually, we find $G_{f^{s+1}(x_0), y_0} \neq 0$, unless contradiction occurs earlier. If so, $f^{s+1}(x_0)$ would have been added to C_{x_0} in our algorithm after adding $f^s(x_0)$. We have contradiction once again.

Therefore, our algorithm and subsequent procedures show that the first nonzero entries (both value and location) of each row uniquely determines $G \in FinOrb(B_n)$. Our algorithm essentially partitions C, any possible subset of $\{1, \dots n\}$, into subsets where each subset has at least two elements — x_j added in Step 2 and $f(x_j) = y_j$ Step 3 for the first time.

Any such partition is possible, as we set $G_{x'_i, x'_{i+1}}$, $i = 1, \dots s$ to be nonzero for any subset $C_{x'_1} = \{x'_1, x'_2, \dots x'_{s+1}\}$ that is a part of partition $C = \bigcup_{i=1}^{p} C_{x_{a_i}} = \{x_1, x_2, \dots x_t\}$, where $a_1, \dots a_p$ are the indexes for numbers in $x_1, x_2, \dots x_t$ that created new subset $C_{x_{a_i}}$ in our algorithm. Furthermore, we may determine, $G_{x'_i, x'_{i+1}}$, $i = 1, \dots s$, in each subset $C_{x'_1} = \{x'_1, x'_2, \dots x'_{s+1}\}$ of C, to be either 2 or -2. This can be done by the binary information of integers $0 \leq \lambda_{x_{a_i}} < 2^{|C_{x_{a_i}}|-1}$ for each $i \in \{1, \dots p\}$. Both the partition itself and the $\lambda_{x_{a_i}}$ for each subset are dependent on the choice of C. Therefore, $G \in FinOrb(B_n)$ with only entries 0 and ± 2 corresponds to a particular partition (where each part has at least two elements) of $C \subseteq \{1, \dots n\}$ plus p numbers in $\mathbb{Z}_{\left(2^{|C_{x_{a_i}}|-1}\right)}$, $i = 1, \dots p$, which is the same as one number in $\mathbb{Z}_{\left(\prod_{i=1}^{p} 2^{|C_{x_{a_i}}|-1}\right)} = \mathbb{Z}_{\left(2^{|C|-p}\right)}$. □

4.1.2 Proof of Theorem 1.3 Part 1

To count all finite orbits of only entries 0 and ± 2, we find a representative of each orbit to analyze.

Definition 4.4. *Consider block matrices in U_+ with the upper-triangle filled with 2.*

$$\begin{bmatrix} 1 & 2 & \cdots & 2 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Then an $n \times n$ segmented matrix $\Gamma(n; (t_1 > \ldots > t_s); (r_1, \ldots r_s)) \in U_+$ is a concatenation of such matrices. The first r_1 blocks of 2's are $(t_1+1) \times (t_1+1)$, the next r_2 are $(t_2+1) \times (t_2+1)$, and so on. If $\sum_{i=1}^{s}(t_i+1)r_i < n$, the final remaining rows and columns are filled with 0. If $\sum_{i=1}^{m}(t_i+1)r_i = n$, Γ is called a compact segmented matrix.

For example, below is $\Gamma(8; (2,1), (1,2))$, which is not compact.

$$\begin{bmatrix} 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Lemma 4.5. *If for any $i < j < k \in \{1, \ldots n\}$, the submatrix of a matrix G*

$$G(i,j,k) = \begin{bmatrix} 1 & G_{ij} & G_{ik} \\ 0 & 1 & G_{jk} \\ 0 & 0 & 1 \end{bmatrix} \in T_1 \cup T_2 \cup \{I\},$$

then $G \in FinOrb(B_n)$.

Proof. Suppose $G \in Orb(\Gamma)$ is such that $G(i,j,k) \in T_1 \cup T_2 \cup \{I\}$ for any $i < j < k \in \{1, \ldots n\}$.

For $i \in \{1, \ldots n-1\}$, σ_i only affects G_{ij}, G_{i+1j} for $j = i+1, \ldots n$ and G_{ji}, G_{ji+1} for $j = 1, \ldots i-1$. The changes in these entries only affect the entry triples (G_{ij}, G_{ik}, G_{jk}), $(G_{i+1j}, G_{i+1k}, G_{jk}), (G_{ji}, G_{kj}, G_{ki}), (G_{ji+1}, G_{kj}, G_{ki+1})$ for any suitable k that puts the triple in the upper-triangle. This is because they are the only triples triples constrained by $G(i,j,k) \in T_1 \cup T_2 \cup \{I\}$. Some casework shows that for any $G \in U_+$ with only entries 0 or ± 2 such that $G(i,j,k) \in T_1 \cup T_2 \cup \{I\}$ for any $i < j < k \in \{1, \ldots n\}$, there is $\sigma_t(G)(i,j,k) \in T_1 \cup T_2 \cup \{I\}$ for $t = 1, \ldots n-1$.

It is only necessary to casework on one of the generators, without loss of generality. For the sake of simplicity, we only show the casework done for $G_{ii+1} = 2$, given a certain i. If $G_{ii+1} = 2$, then $G_{ij} = G_{i+1j} = 0, \pm 2$ for any $j > i$, because $G(i, i+1, j) \in T_1 \cup T_2 \cup \{I\}$.

If $G_{ij} = G_{i+1j} = 0$, then σ_i preserves the values of these two entries.

If $G_{ij} = G_{i+1j} = 2$, we have cases $G_{ik} = G_{i+1k} = G_{jk} = \pm 2$ and $G_{ik} = G_{i+1k} = G_{jk} = 0$. The first case gives $\sigma_i(G)_{ij} = -2, \sigma(G)_{ik} = \mp 2, \sigma(G)_{jk} = \pm 2$ and $\sigma_i(G)_{i+1j} = 2, \sigma(G)_{i+1k} = \pm 2, \sigma(G)_{jk} = \pm 2$; the second case gives $\sigma_i(G)_{ij} = -2, \sigma(G)_{ik} = \sigma(G)_{jk} = 0$ and $\sigma_i(G)_{i+1j} = 2, \sigma(G)_{i+1k} = \sigma(G)_{jk} = 0$. In any event, $G(i,j,k) \in T_1 \cup T_2 \cup \{I\}$ is satisfied.

If $G_{ij} = G_{i+1j} = -2$, we have cases $G_{ik} = G_{i+1k} = -G_{jk} = \pm 2$ and $G_{ik} = G_{i+1k} = G_{jk} = 0$. The first case gives $\sigma_i(G)_{ij} = 2, \sigma(G)_{ik} = \pm 2, \sigma(G)_{jk} = \pm 2$ and $\sigma_i(G)_{i+1j} = -2, \sigma(G)_{i+1k} = \mp 2, \sigma(G)_{jk} = \pm 2$; the second case gives $\sigma_i(G)_{ij} = 2, \sigma(G)_{ik} = \sigma(G)_{jk} = 0$ and $\sigma_i(G)_{i+1j} = -2, \sigma(G)_{i+1k} = \sigma(G)_{jk} = 0$. In any event, $G(i,j,k) \in T_1 \cup T_2 \cup \{I\}$ is satisfied.

After the complete casework, we know that if $G(i,j,k) \in T_1 \cup T_2 \cup \{I\}$ for any $i < j < k \in \{1, \ldots n\}$, then $\sigma_t(G)(i,j,k) \in T_1 \cup T_2 \cup \{I\}, t = 1, \ldots n-1$, and hence $\beta(G)(i,j,k) \in T_1 \cup T_2 \cup \{I\}, \forall \beta \in B_n$ as we repetitively apply the condition. This implies that the orbit generated by G such that $G(i,j,k) \in T_1 \cup T_2 \cup \{I\}$ will only have matrix entries 0 or ± 2. Since matrices with entries $0, \pm 2$ form a finite set and $\Gamma(i,j,k) \in T_1 \cup T_2 \cup \{I\}$, we have $\Gamma \in FinOrb(B_n)$. □

Obviously, segmented matrices satisfy the condition in Lemma 4.5. As a immediate consequence, we have

Proposition 4.6. *Each segmented matrix generates a finite orbit of B_n.*

Next, Proposition 4.3 helps us prove that

Proposition 4.7. *(1) Segmented matrices generate all finite orbits with matrices that only include entries of 0 and ± 2.*

(2) Distinct segmented matrices generate distinct orbits.

Proof.

Lemma 4.8. *A matrix $G \in FinOrb(B_n)$ in the form of*

$$\begin{bmatrix} 1 & 2\delta_{12} & \cdots & 2\delta_{1n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2\delta_{n-1n} \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

where $\delta_{ij} = \pm 1$ *and* $i, j \in \{1, \dots n\}$, *is in the same orbit as matrix*

$$\begin{bmatrix} 1 & 2 & \cdots & 2 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Proof. For given $i \in \{1, \dots n\}$: i. If $G_{ii+1} = 2$ and $G_{i+1,i+2} = -2$, we have $G_{ii+2} = -2$ by Lemma 4.2. $G' = \sigma_{i+1}(G)$ has $G'_{ii+1} = G_{i+1,i+2} = 2$. Let $\mu_i(2, -2) = \sigma_i$ for $i = 1, \dots n-1$.

ii. If $G_{ii+1} = -2$ and $G_{i+1,i+2} = 2$, we have $G_{ii+2} = -2$ by Lemma 4.2. $G' = \sigma^2_{i+1}(G)$ has $G'_{ii+1} = G_{i+1,i+2} = 2$. Let $\mu_i(-2, 2) = \sigma_i^2$ for $i = 1, \dots n-1$.

iii. If $G_{ii+1} = -2 = G_{i+1,i+2} = -2$, we have $G_{ii+2} = 2$ by Lemma 4.2. $G' = \sigma^3_{i+1}(G)$ has $G'_{ii+1} = G_{i+1,i+2} = 2$. Let $\mu_i(-2, -2) = \sigma_i^3$ for $i = 1, \dots n-1$.

Let $G = G^{(0)}$. We define the following recursive formula for $i = 1, \dots \lfloor \frac{n-1}{2} \rfloor$:

$$G^{(i)} = \mu_{2i}(G^{(i-1)}_{2i-1,2i}, G^{(i-1)}_{2i,2i+1})(G^{(i-1)}).$$

If n is odd, let $\Gamma' = G^{\left(\frac{n-1}{2}\right)}$. If n is even, let $\Gamma' = \sigma_{n-1}(G^{\left(\frac{n-1}{2}\right)})$.

Since any power of σ_i, $i \in \{3, \dots n-1\}$ acting on G preserves the values of $G_{12}, \dots G_{i-2,i-1}$, Γ' is a matrix where the sub-diagonal is entirely 2. Due to Lemma 4.2, the entire upper-triangle would have to be filled with 2 in order to have $G \in FinOrb(B_n)$. □

Given $G \in FinOrb(B_n)$, we divide the first entry of each row into sets $C_1, \dots C_k$ as described in Proposition 4.3. Consider $C_1 = \{x_1, x_2, \dots x_{s+1}\}$ (Recall that C_1 is a set of matrix subscripts in nature). Let

$$\beta(i, j, k) = \sigma_{k-3}\sigma_{k-4} \cdots \sigma_i \sigma_{k-2}\sigma_{k-3} \cdots \sigma_j$$

given certain $i, j, k \in \{1, \dots n\}, i < j < k$. The matrix

$$G' := \left(\prod_{i=1}^{n-2} \beta(x_i, x_{i+1}, x_{i+2}) \right) (G)$$

, where the product goes from right to left as i increases, gives $G'_{ii+1} = G_{x_i,x_{i+1}}$ ($i = \{1, 2, \dots s\}$), as the braids worked the same way as the ones in Lemma 4.2 to send M_{ij}, M_{jk} to the sub-diagonal. Using the logic in the proof of Proposition 4.3, these nonzero sub-diagonal elements determine unique values for G'_{ij} ($i < j, i, j \in \{1, \dots s\}$). No other nonzero entries should exist in the first s rows of G', for the same reason as the proof by contradiction that uses (29), (30). Now we narrow down to the $(n-s) \times (n-s)$ submatrix that excludes the first s rows and columns of G'.

Using the same idea on $C_2, \dots C_k$ and apply more braid operators, we obtain a matrix that has all its nonzero entries in the locations of nonzero entries of a segmented matrix.

Applying the process described in Lemma 4.8 onto each block of ± 2's, we finally get a segmented matrix. Thus, all $G \in FinOrb(B_n)$ with only the entries 0 and ± 2 belong to $Orb(\Gamma)$ for a certain segmented matrix Γ.

Now we prove Part (2) of the proposition. For any matrix G in the orbit of a segmented matrix, σ_i preserves the locations of all non-zero entries of G if $G_{ii+1} = \pm 2$, because the entries G_{ij}, G_{i+1j} (or G_{ji}, G_{ji+1}) must be either both zero or both non-zero by Lemma 4.2. For instance,

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\sigma_4} \begin{bmatrix} 1 & -2 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & -2 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

If $G_{ii+1} = 0$, then the non-zero entries on the $i-th$ and $i+1$-th columns switch columns, and the non-zero entries on the i-th and $i+1$-th rows switch rows. For instance,

$$\begin{bmatrix} 1 & -2 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\sigma_3} \begin{bmatrix} 1 & -2 & 2 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Because of this, the numbers of non-zero entries on the different rows (or columns) form a constant group of numbers. In other words, braid actions only switches around the number of non-zero entries of each row (and column). Since each segmented matrix has a distinct group of numbers for the count of non-zero entries on each row, they must generate distinct orbits. □

If the first $m = \sum_{i=1}^{s}(t_i+1)r_i$ rows/columns of a segmented matrix forms a compact segmented matrix, it is a concatenation of $d = \sum_{i=1}^{s} r_i$ blocks of submatrices with the upper-triangle filled with 2. We have essentially described Γ as partition of $m - d = \sum_{i=1}^{s} t_i r_i$ into d positive integers. The d integers include r_i copies of t_i for $i = 1, \dots s$. Here, m can be any number smaller or equal to n, and d can be any number smaller or equal to $\lfloor \frac{m}{2} \rfloor$.

Since each partition of $m - d$ into d parts forms a distinct segmented matrix and hence a distinct finite orbit by Proposition 4.7, we have the double sum for the total number of orbits with only the entries 0 and ± 2 expressed in Theorem 1.3. We add 1 to include the trivial finite orbit $\{I\}$.

4.1.3 Proof of Theorem 1.3 Part 2

Fitting $\Gamma(n;(t_1,\dots t_s);(r_1,\dots r_s))$ into the context of Proposition 4.3, we have set partition $C=\bigcup_{i=1}^{k} C_i$ for Γ, where $|C_1|=\dots=|C_{r_1}|=t_1+1$, $|C_{r_1+1}|=\dots=|C_{r_1+r_2}|=t_2,\dots$ We also require $|C|=m$.

Under these conditions, we have

$$\frac{n!}{m!(n-m)!} \tag{31}$$

ways to choose C. We also have

$$\frac{m!}{\prod_{i=1}^{s} r_i!\cdot((t_i+1)!)^{r_i}} \tag{32}$$

ways to partition C such that each subset has at least two elements. $(t_i+1)!$ represents all the permutations of elements within the same partitioned set, which we don't distinguish; $r_i!$ represents all the permutations of partitioned sets with the same cardinality, which we also don't distinguish when counting partitions.

Finally, we may choose integers $0\le\lambda_{ij}<2^{t_i}$ $(i=1,\dots s; j=1,\dots r_i)$ for all partitioned sets of C. That adds up to

$$\prod_{i=1}^{s}(2^{t_i})^{r_i}=\prod_{i=1}^{s}2^{t_i r_i} \tag{33}$$

possibilities. The product of (31),(32),(33) gives the expression in Part 2 of Theorem 1.3.

4.2 Preparation for computing certain 5×5 finite orbits

Proposition 4.9. *For $M\in FinOrb(B_n)$, if for any $\beta\in B_n$ and $M'=\beta(M)$, the submatrix*

$$M_i=\begin{bmatrix}1 & M_{i-1i} & M_{i-1i+1}\\ 0 & 1 & M_{ii+1}\\ 0 & 0 & 1\end{bmatrix}$$

has $M_i\in FinOrb(B_3)$, $M_i\notin FinOrb(S_n)$, and $|M_i+M_i^T|\neq 0$ for $i=2,\dots n-1$, then any entry of M is in the set $\{2\cos(\frac{p\pi}{q})\mid q=2,3,4,5; p=1,2,\dots q-1\}$.

Proof. Dubrovin and Mazzocco have proven this proposition for 3×3 matrix M in [4], but under slightly different definitions for the action of B_n on matrices. We adapt their proof to our definition.

Let $x_1 = M_{12}$, $x_2 = M_{13}$, $x_3 = M_{23}$. The braid action on coordinate (x_1, x_2, x_3) is the following

$$\sigma_1(x_1, x_2, x_3) = (-x_1, x_3 - x_1x_2, x_2)$$

$$\sigma_2(x_1, x_2, x_3) = (x_2 - x_1x_3, x_1, -x_3)$$

The triple is essentially generated under σ_1 and the cyclic permutation $s : (x_1, x_2, x_3) \to (x_3, x_1, x_2)$, where $\sigma_2 = s \cdot \sigma_1 \cdot s^2$. Thus, it suffices to consider the cyclic permutations of

$$\sigma : (x_i, x_j, x_k) \to (-x_i, x_k - x_ix_j, x_j)$$

By Theorem 1.1, suppose (x_1, x_2, x_3) generates a finite set under the braid group, let $x_i = -2\cos(\pi r_i) = 2\cos(\pi(1 - r_i))$. σ acts on (r_i, r_j, r_k) as

$$\sigma : (r_i, r_j, r_k) \to (1 - r_i, r'_k, r_j)$$

where $0 < r_k < 1$ is a rational number such that

$$\cos(\pi r'_k) = \cos(\pi r_k) + 2\cos(\pi r_i)\cos(\pi r_j)$$

Hence,

$$\cos(\pi r_k) + \cos(\pi r_i + \pi r_j)) + \cos(\pi r_i - \pi r_j) + \cos(\pi - \pi r'_k) = 0$$

From here, [4] obtains the only triples (r_i, r_j, r_k) that satisfy the equation above. They correspond to the following the five orbits generated by (x_1, x_2, x_3)

$$(2\cos\frac{\pi}{2}, 2\cos\frac{2\pi}{3}, 2\cos\frac{2\pi}{3})$$

$$(2\cos\frac{\pi}{2}, 2\cos\frac{2\pi}{3}, 2\cos\frac{3\pi}{4})$$

$$(2\cos\frac{\pi}{2}, 2\cos\frac{2\pi}{3}, 2\cos\frac{4\pi}{5})$$

$$(2\cos\frac{\pi}{2}, 2\cos\frac{2\pi}{3}, 2\cos\frac{3\pi}{5})$$

$$(2\cos\frac{\pi}{2}, 2\cos\frac{4\pi}{5}, 2\cos\frac{3\pi}{5})$$

Their orbit lengths are 16, 44, 40, 40, 80, respectively. Our proof is done for 3×3 matrices. Note that our orbit lengths are different from the lengths stated in [4], because we used a slightly different definition for the braid group action on matrices.

For an $n \times n$ matrix that is finite under B_n, the submatrix

$$M(i-1, i, i+1) = \begin{bmatrix} 1 & M_{i-1i} & M_{i-1i+1} \\ 0 & 1 & M_{ii+1} \\ 0 & 0 & 1 \end{bmatrix}$$

must be within $FinOrb(B_3)$. So, the entries of the sub-diagonal $M_{12}, \dots M_{n-1n}$ are already in the set $\{2\cos(\frac{p\pi}{q}) \mid q = 2,3,4,5; p = 1,2,\dots q-1\}$.

Suppose the lemma is true for an $(n-1)\times(n-1)$ matrix. Extending the n-th column and row leaves us with the entries $M_{1n}, M_{2n}, \dots M_{n-2}$ to check. For $M_{in}, i \in \{1,\dots n-2\}$, let $M' = \sigma_{n-3}\sigma_{n-4}\cdots\sigma_i(M)$. We have $M_{i,n-1} = M'_{n-2,n-1}, M_{in} = M'_{n-2,n}, M_{n-1,n} = M'_{n-1,n}$. The 3×3 submatrix

$$\begin{bmatrix} 1 & M'_{n-2,n-1} & M_{n-2,n} \\ 0 & 1 & M_{n-1,n} \\ 0 & 0 & 1 \end{bmatrix}$$

must belong to $FinOrb(B_3)$. Therefore, $M_{i,n} = M'_{n-2,n}$ must belong to the finite set described earlier. □

To present more information for future research, in our computer algorithm for 5×5 matrices, we also allowed any entry of M to be ± 2, expanding the set to be $\{2\cos(\frac{p\pi}{q}) \mid q = 1,2,3,4,5; p \in \mathbb{Z}\}$.

4.3 Preparation for computing all 4×4 finite orbits: Proof of theorem 1.5

Suppose there exists an entry $M_{ij} = 2\cos(\frac{p\pi}{q})$ $(1 \le i < j \le 4)$, where $q \in \{1,2,3,4,5,6,8,9,10,12,15,16,20,25\}, p \in \mathbb{Z}$. The braid transformation

$$M' = \sigma_2\cdots\sigma_i\sigma_3\cdots\sigma_j(M)$$

gives $M'_{34} = M_{ij}$.

Regardless of what the values of M'_{23} and M'_{24} are, because $M'_{34} \notin \{2\cos(\frac{p\pi}{q}) \mid q = 1,2,3,4,5,6,8,9,10,12,15,16,20,25; p \in \mathbb{Z}\}$ and Lemma 4.2 and Proposition 4.9 holds, the matrices

$$\begin{bmatrix} 1 & M'_{23} & M'_{24} \\ 0 & 1 & M'_{34} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & M'_{13} & M'_{14} \\ 0 & 1 & M'_{34} \\ 0 & 0 & 1 \end{bmatrix} \tag{34}$$

generate degenerate orbits. Recall that we may express the two submatrices as $M(2,3,4), M(1,3,4)$, respectively.

Let p_{ij}, q_{ij} $(i,j = 1,2,3,4; i < j)$ be integers such that $2\cos(\frac{p_{ij}}{q_{ij}}\pi) = M'_{ij}$, and $\gcd(p_{ij}, q_{ij}) = 1$.

If $q_{23} \in \{1,2,3,4,5\}$,

$$\frac{p_{23}}{q_{23}} + \frac{p_{34}}{q_{34}} = \frac{p_{23}q_{34} + p_{34}q_{23}}{q_{23}q_{34}} = \frac{p_{24}}{q_{24}}.$$

We rewrite this with

$$\begin{aligned}
p_{23} &= \gcd(p_{23}, p_{34}) p'_{23}, \\
p_{34} &= \gcd(p_{23}, p_{34}) p'_{34}, \\
q_{23} &= \gcd(q_{23}, q_{34}) q'_{23}, \\
q_{34} &= \gcd(q_{23}, q_{34}) q'_{34},
\end{aligned}$$

yielding

$$\frac{(p'_{23}q'_{34} + p'_{34}q'_{23}) \cdot \gcd(p_{23}, p_{34})}{q'_{23}q'_{34} \cdot \gcd(q_{23}, q_{34})} = \frac{p_{24}}{q_{24}}$$

where $\gcd(p'_{23}, p'_{24}) = \gcd(q'_{23}, q'_{24}) = \gcd(p'_{23}, q'_{23}) = \gcd(p'_{24}, p'_{24}) = 1$. Thus, $\gcd(p'_{23}q'_{34} + p'_{34}q'_{23}, q'_{23}q'_{34}) = 1$ and $\gcd(\gcd(p_{23}, p_{34}), q'_{23}q'_{34}) = 1$. Hence, $q'_{23}q'_{34} | q_{24}$.

Knowing that $q_{34} \notin \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 20, 25\}$ and $q_{23} \in \{1, 2, 3, 4, 5\}$, we have $q_{24} \geq q'_{23}q'_{34} > 5$, which by Proposition 4.9, shows that the matrix

$$\begin{bmatrix} 1 & M'_{12} & M'_{14} \\ 0 & 1 & M'_{24} \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

generates a degenerate orbit.

Set $r_{ij} = \frac{p_{ij}}{q_{ij}}$. So far we know from (34) and (35) that

$$\begin{aligned}
r_{34} + r_{23} &= r_{24} \\
r_{34} + r_{13} &= r_{14} \\
r_{12} + r_{24} &= r_{14}
\end{aligned}$$

, which implies that $r_{12} + r_{23} = r_{13}$.

Therefore, the matrix $M'(1, 2, 3)$ also generates a degenerate orbit. Given that all 3×3 submatrices now generate degenerate orbits, [7] tells us that M' itself generates a degenerate orbit.

On the other hand, if $q_{23} \notin \{1, 2, 3, 4, 5\}$, the deduction is similar. Apparently, $M'(1, 2, 3)$ generates a degenerate orbit, which helps us show that $r_{12} + r_{24} = r_{14}$ and we arrive at the same conclusion. This ends the proof.

With Theorem 1.5, we only require a search algorithm to look for all finite orbits where matrices only have entries within the finite set $M_{ij} \notin \{2\cos(\frac{p\pi}{q}) \mid q = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 20, 25; p \in \mathbb{Z}\}$ $(1 \leq i < j \leq 4)$. It turns out that all entries of $M \in FinOrb(B_4)$ for a non-degenerate, non-S_n orbit are within the smaller finite set of $M_{ij} \notin \{2\cos(\frac{p\pi}{q}) \mid q = 1, 2, 3, 4, 5; p \in \mathbb{Z}\}$. See Table 6.

4.4 Computed Finite Orbits

We program an algorithm to find all finite orbits, given a finite set A containing values that all entries of the orbit must take. For instance, Conjecture 1.4 demands the set $A = \{0, 1, -1\}$, Section 4.2 demands $A = \{2\cos(\frac{p\pi}{q}) \mid q = 1, 2, 3, 4, 5; p \in \mathbb{Z}\}$, and Section 4.3 demands $A = \{2\cos(\frac{p\pi}{q}) \mid q = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 20, 25; p \in \mathbb{Z}\}$.

Given A, for each out of the $|A|^{\frac{(n-1)n}{2}}$ numerated matrices, the algorithm searches along a tree with outdegree $n-1$, since there are $n-1$ possible generators of B_n. While searching, the program parses every new matrix it visits into an $\frac{(n-1)n}{2}$-digit hexadecimal, where each digit represents an entry from the matrix (there are only 11 possible values). It stores the parsed hexadecimal into a boolean map as the key and sets the value to True. Whenever the program visits a matrix that is already visited according to the map, it no longer branches off along that matrix. The particular order of searching is Breadth-first search (BFS), to optimize space and runtime.

For this research, all computations are done on a Tencent Cloud server. C++ and Python programs (primarily search algorithms for finite orbits) are run on a 20-core 80GB NVIDIA Tesla-T4.

In the following tables, a representative of a matrix in U_+ is described with a sequence $\phi_1, \phi_2, \ldots \phi_{\frac{n(n-1)}{2}}$: the upper-triangular entries of the matrix, read left-to-right and top-to-bottom, is the sequence $2\cos\phi_1, 2\cos\phi_2, \ldots 2\cos\phi_{\frac{n(n-1)}{2}}$. For instance, the sequence $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{2}$ represents the matrix

$$\begin{bmatrix} 1 & 2\cos(\frac{2\pi}{3}) & 2\cos(\frac{2\pi}{3}) & 2\cos(\frac{\pi}{3}) \\ 0 & 1 & 2\cos(\frac{\pi}{5}) & 2\cos(\frac{2\pi}{5}) \\ 0 & 0 & 1 & 2\cos(\frac{\pi}{2}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Length	Representative
32	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$
64	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
72	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}$
200	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}$

Table 1: All non-degenerate non-S_n 4×4 finite orbits with only the entries 0, ± 1

Length	Representative
160	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
160	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
360	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1000	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
3456	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
3456	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
4096	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$

Table 2: All non-degenerate non-S_n 5×5 finite orbits with only the entries 0, ± 1

Length	Representative
320	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
480	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
960	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
1080	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1920	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
2160	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
2560	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
3000	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
6000	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
20736	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
20736	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
24576	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
34560	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
38880	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
76832	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
81920	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
100000	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
110592	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
209952	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$

Table 3: All non-degenerate non-S_n 6×6 finite orbits with only the entries 0, ± 1

Length	Representative
560	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1120	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
2520	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
6720	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
6720	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
6720	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
7000	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
15120	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
17920	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
17920	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
40320	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
42000	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
72576	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
72576	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
86016	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
112000	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
145152	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
145152	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
172032	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
241920	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
272160	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
362880	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
537824	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
573440	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
700000	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
774144	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1469664	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
2097152	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
2211840	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
2400000	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
2985984	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
7558272	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
9953280	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
10800000	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
15059072	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$

Table 4: All non-degenerate non-S_n 7×7 finite orbits with only the entries 0, ± 1

Length	Representative
4	$\pi,\ \pi,\ 0$
16	$\frac{2\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{3}$
16	$\pi,\ \frac{2\pi}{3},\ \frac{\pi}{3}$
40	$\frac{4\pi}{5},\ \frac{4\pi}{5},\ \frac{\pi}{3}$
40	$\frac{3\pi}{5},\ \frac{3\pi}{5},\ \frac{\pi}{3}$
44	$\frac{3\pi}{4},\ \frac{3\pi}{4},\ \frac{\pi}{3}$
48	$\frac{3\pi}{5},\ \frac{2\pi}{5},\ \frac{\pi}{5}$
80	$\frac{3\pi}{5},\ \frac{2\pi}{3},\ \frac{\pi}{5}$

Table 5: All non-degenerate non-S_n 3×3 finite orbits

Length	Representative
8	$\pi,\ \pi,\ \pi,\ 0,\ 0,\ 0$
16	$\pi,\ \pi,\ \frac{\pi}{2},\ 0,\ \frac{\pi}{2},\ \frac{\pi}{2}$
32	$\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{3},\ 0,\ \frac{\pi}{3}$
32	$\frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{3}$
64	$\frac{2\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{2}$
64	$\pi,\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{2}$
72	$\frac{2\pi}{3},\ \frac{2\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{2}$
72	$\pi,\ \pi,\ \frac{2\pi}{3},\ 0,\ \frac{\pi}{3},\ \frac{\pi}{3}$
80	$\frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{3\pi}{5},\ \frac{2\pi}{5},\ \frac{2\pi}{5},\ \frac{\pi}{3}$
80	$\frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{4\pi}{5},\ \frac{\pi}{5},\ \frac{\pi}{5},\ \frac{\pi}{3}$
96	$\frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{2}$
96	$\frac{4\pi}{5},\ \frac{3\pi}{5},\ \frac{4\pi}{5},\ \frac{\pi}{5},\ 0,\ \frac{\pi}{5}$
144	$\frac{\pi}{2},\ 0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \frac{\pi}{2},\ \frac{\pi}{4}$
160	$\frac{4\pi}{5},\ \frac{4\pi}{5},\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{2}$
160	$\frac{3\pi}{5},\ \frac{3\pi}{5},\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{2}$
176	$\frac{3\pi}{4},\ \frac{3\pi}{4},\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{2}$
184	$\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{3\pi}{5},\ \frac{2\pi}{5},\ \frac{\pi}{5},\ \frac{\pi}{3}$

Length	Representative
192	$\frac{3\pi}{5},\ \frac{2\pi}{5},\ \frac{\pi}{2},\ \frac{\pi}{5},\ \frac{\pi}{2},\ \frac{\pi}{2}$
200	$\frac{3\pi}{5},\ \frac{3\pi}{5},\ \frac{2\pi}{5},\ 0,\ \frac{\pi}{5},\ \frac{\pi}{5}$
200	$\pi,\ \frac{3\pi}{5},\ \frac{4\pi}{5},\ \frac{2\pi}{5},\ \frac{\pi}{5},\ \frac{\pi}{5}$
200	$\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{2}$
216	$\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{2}$
312	$\frac{3\pi}{5},\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{5},\ \frac{\pi}{2},\ \frac{\pi}{2}$
720	$\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{3\pi}{5},\ \frac{\pi}{2},\ \frac{2\pi}{5},\ \frac{\pi}{2}$
720	$\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{4\pi}{5},\ \frac{\pi}{2},\ \frac{\pi}{5},\ \frac{\pi}{2}$
864	$\frac{2\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{5},\ \frac{2\pi}{5},\ \frac{\pi}{2}$
1224	$\frac{4\pi}{5},\ \frac{2\pi}{5},\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{5},\ \frac{\pi}{2}$
1224	$\frac{3\pi}{5},\ \frac{\pi}{5},\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{5},\ \frac{\pi}{2}$
1338	$\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{4\pi}{5},\ \frac{\pi}{3},\ \frac{\pi}{5},\ \frac{\pi}{2}$
1338	$\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{3\pi}{5},\ \frac{\pi}{3},\ \frac{2\pi}{5},\ \frac{\pi}{2}$
1418	$\frac{3\pi}{5},\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{\pi}{5},\ \frac{\pi}{2}$
1418	$\frac{4\pi}{5},\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{5},\ \frac{\pi}{2}$
1680	$\frac{3\pi}{5},\ \frac{2\pi}{3},\ \frac{3\pi}{5},\ \frac{\pi}{5},\ \frac{\pi}{3},\ \frac{\pi}{2}$
2432	$\frac{4\pi}{5},\ \frac{3\pi}{5},\ \frac{4\pi}{5},\ \frac{\pi}{5},\ \frac{\pi}{3},\ \frac{\pi}{2}$
2580	$\frac{4\pi}{5},\ \frac{3\pi}{5},\ \frac{4\pi}{5},\ \frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{2}$

Table 6: All non-degenerate non-S_n 4×4 finite orbits

Length	Representative	Length	Representative
16	$\pi, \pi, \pi, \pi, 0, 0, 0, 0, 0, 0$	480	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
40	$\pi, \pi, \pi, \frac{\pi}{2}, 0, 0, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}$	480	$\frac{4\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{\pi}{2}, \frac{\pi}{5}, 0, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$
40	$\pi, \pi, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	480	$\frac{3\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
80	$\frac{\pi}{2}, 0, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, 0, \frac{\pi}{2}, \frac{\pi}{2}$	640	$\pi, \pi, \pi, \frac{2\pi}{3}, 0, 0, \frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{3}$
80	$\pi, \frac{\pi}{2}, \frac{\pi}{2}, \pi, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$	640	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
80	$\pi, \frac{\pi}{2}, \frac{\pi}{2}, \pi, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$	720	$\frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}$
80	$\pi, \frac{\pi}{2}, \frac{\pi}{2}, \pi, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}$	720	$\frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$
80	$\pi, \frac{\pi}{2}, \frac{\pi}{2}, \pi, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	720	$\frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$
160	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, 0, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	720	$\frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, 0, 0, \frac{\pi}{2}, \frac{\pi}{2}$
160	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	768	$\frac{3\pi}{5}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
160	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
160	$\pi, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	800	$\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	880	$\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
320	$\frac{\pi}{2}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	880	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
360	$\frac{\pi}{2}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}$	880	$\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
360	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	880	$\frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
360	$\pi, \pi, \frac{2\pi}{3}, \frac{\pi}{2}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	880	$\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
400	$\pi, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{2}, 0, \frac{\pi}{2}$	888	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{5}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
400	$\frac{4\pi}{5}, \frac{4\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	960	$\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$
400	$\frac{3\pi}{5}, \frac{3\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	960	$\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$
400	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{4\pi}{5}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	960	$\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$
400	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{5}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	960	$\frac{3\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
440	$\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	960	$\frac{3\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}$

Table 7: All non-degenerate non-S_n 5×5 finite orbits with entries in $\{2\cos(\frac{p\pi}{k}) \mid k = 1, 2, 3, 4, 5; p \in \mathbb{Z}\}$ (Part 1)

Length	Representative	Length	Representative
1000	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	4992	$\pi, \pi, \frac{3\pi}{5}, \frac{4\pi}{5}, 0, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{5}$
1000	$\pi, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$	5976	$\frac{4\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1000	$\frac{3\pi}{5}, \frac{3\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{2}, 0, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}$	5976	$\frac{3\pi}{5}, \frac{\pi}{5}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1080	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	6552	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{5}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1536	$\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	6552	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{5}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1536	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	6952	$\frac{4\pi}{5}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1536	$\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	6952	$\frac{3\pi}{5}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1536	$\frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	8256	$\frac{3\pi}{5}, \frac{2\pi}{3}, \frac{3\pi}{5}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1536	$\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	11904	$\frac{4\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
1920	$\pi, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, 0, \frac{\pi}{2}, \frac{\pi}{2}$	12624	$\frac{4\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
2560	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, 0, \frac{\pi}{2}, \frac{\pi}{2}$	48136	$\frac{4\pi}{5}, \frac{3\pi}{5}, \frac{2\pi}{3}, \frac{4\pi}{5}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
3456	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	48136	$\frac{3\pi}{5}, \frac{\pi}{5}, \frac{2\pi}{3}, \frac{3\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
3456	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	55728	$\pi, \frac{4\pi}{5}, \frac{4\pi}{5}, \frac{2\pi}{3}, \frac{\pi}{5}, \frac{\pi}{5}, \frac{\pi}{3}, 0, \frac{\pi}{2}, \frac{\pi}{2}$
3456	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	81328	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{\pi}{3}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
3600	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	168208	$\frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
3600	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	168208	$\frac{2\pi}{3}, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
3648	$\pi, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{2\pi}{3}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	211975	$\frac{3\pi}{5}, \frac{\pi}{3}, \frac{3\pi}{5}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
3840	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	211975	$\frac{2\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
4096	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	235136	$\frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$
4096	$\frac{\pi}{2}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$	404696	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{5}, \frac{2\pi}{3}, \frac{2\pi}{5}, \frac{\pi}{5}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}$
4176	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$	468254	$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$

Table 8: All non-degenerate non-S_n 5×5 finite orbits with entries in $\{2\cos(\frac{p\pi}{k}) \mid k = 1, 2, 3, 4, 5; p \in \mathbb{Z}\}$ (Part 2)

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