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**RESEARCH REPORT**

**The Team**

Name of team member: Akshaya Chakravarthy

School: Thomas Jefferson High School for Science and Technology

City, Country: Fairfax, Virginia, U.S.A.

Name of supervising teacher: Agustina Czenky

Job Title: PhD Student

School/Institution: University of Oregon

City, Country: Eugene, Oregon, U.S.A.

Name of supervising teacher: Julia Plavnik

Job Title: Associate Professor

School/Institution: Indiana University

City, Country: Bloomington, Indiana, U.S.A.

**Title of Research Report**

On Modular Categories with Frobenius-Perron Dimension Congruent to 2 Mod 4

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AKSHAYA CHAKRAVARTHY

Abstract. A *modular tensor category* is a fusion category with additional structure that has applications in other fields of mathematics as well as quantum physics. We contribute to the classification of modular categories $\mathcal{C}$ with $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$. We prove that such categories factorize as $\mathcal{C} \cong \widetilde{\mathcal{C}} \boxtimes \text{semion}$, where $\widetilde{\mathcal{C}}$ is an odd-dimensional modular category and semion is the rank 2 pointed modular category. This reduces the classification of these categories to the classification of odd-dimensional modular categories. It follows that modular categories $\mathcal{C}$ with $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$ of rank up to 46 are pointed. We then provide general results about the classification of modular categories, which could lead to a better understanding of the odd-dimensional ones. Lastly, we provide two algorithms and the code for computing Frobenius-Perron dimensions of simple objects in a modular category of dimension not divisible by 4.

**Keywords:** equivariantization, Frobenius-Perron dimension, modular tensor category, rank, subcategory.

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Signature:

Name of supervising teacher: Agustina Czenky

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The participating team declares that the paper submitted is comprised of original research and results obtained under the guidance of the instructor. To the team's best knowledge, the paper does not contain research results, published or not, from a person who is not a team member, except for the content listed in the references and the acknowledgment. If there is any misinformation, we are willing to take all the related responsibilities.

Name of Team Member: Akshaya Chakravarthy

Signature of team member:

Name of instructor: Agustina Czenky

Signature of the instructor:

Name of instructor: Julia Plavnik

Signature of the instructor:

Date: August 20, 2023

## Contents

## 1. Introduction

We study modular tensor categories (MTCs), which have several applications in both, mathematics and physics. For example, in terms of topological quantum field theory (TQFT), MTCs help understand invariants of closed oriented 3-manifolds as well as generate 3-dimensional TQFTs [T]. In [BK], it was shown that the ideas of MTCs, 3-dimensional TQFTs, and 2-dimensional modular functors are equivalent; thus, classifying MTCs can lead to classification of seemingly unrelated concepts. As stated in [R], anyons, topological quantum fields behaving as finite energy particle-like excitations in topological fields of matter, can be modelled by modular categories. In fact, modular data provides a tool for distinguishing anyon models from each other. Thus, MTCs provide insight into other areas of mathematics and physics.

Formally, an MTC is a fusion category satisfying additional properties. In particular, they have additional braiding and ribbon structures satisfying a non-degeneracy condition for the braiding, see [ENO1, T].

**Example 1.1.** The category of representations of a quantum double of a finite group $G$ is an MTC; in particular, this is equivalent to the category of $G$-equivariant vector bundles on $G$ [BK, Chapter 3].

It was shown in [BNRW1] that there are finitely many (up to equivalence) MTCs of a fixed rank, i.e. number of simple objects. Thus, a natural question is to classify MTCs with a given rank, and many efforts have been made in this direction, see for example [BNRW2, BR, CGP, CP, HR, RSW]. An important class of MTCs are the integral ones, that is, those in which all simple objects have integral Frobenius-Perron dimension, since they are in correspondence with the categories of representations of modular finite-dimensional semisimple quasi-Hopf algebras. Recently, a classification of integral MTCs up to rank 12 was reported in [ABPP], and of odd-dimensional ones up to rank 23 in [BR, CP, CGP].

This paper is dedicated to the study of MTCs with Frobenius-Perron dimension congruent to 2 modulo 4, which are known to be integral by [DN, Corollary 3.2]. They have a special advantage in regards to classification: it has been proven that they cannot be perfect [BP, Corollary 10.11], that is, they have at least one non-trivial invertible object. Moreover, the square of the Frobenius-Perron dimension of any simple object divides the Frobenius-Perron dimension of the category [EG, Lemma 1.2], and thus the Frobenius-Perron dimensions of all simple objects must be odd. This naturally raises the question of how MTCs of dimension congruent to 2 modulo 4 and odd-dimensional MTCs are related, and this paper shows that all MTCs of dimension congruent to 2 modulo 4 can be classified in terms of the odd-dimensional ones.

In order to further classify odd-dimensional MTCs of a fixed rank, this paper adapted and produced several techniques for the classification of MTCs, which can be found in Section 4.

The following is our main result, which will be proven in Section 3. To prove this, we will first prove that the claim holds when $|\mathcal{G}(\mathcal{C})|$ is even, and use this result as well as equivariantization results to show that $|\mathcal{G}(\mathcal{C})|$ cannot be odd.

**Theorem 3.9.** *Let $\mathcal{C}$ be an MTC with $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$. Then, $\mathcal{C} \cong \widetilde{\mathcal{C}} \boxtimes \mathrm{semion}$, where $\widetilde{\mathcal{C}}$ is an odd-dimensional modular category and* semion *is the rank 2 pointed modular category.*

It has been proven that MTCs with dimension congruent to 2 modulo 4 up to rank 10 are pointed [ABPP, Theorem 1.2]. Pointed MTCs are those in which all simple objects have Frobenius-Perron dimension equal to 1, and they are classified by pairs $(G, q)$, where $G$ is a finite abelian group, and $q : G \to \mathbf{k}^\times$ is a non-degenerate quadratic form on $G$, see [EGNO, Example 8.13.5]. As a corollary of the above theorem, we extend the result of [ABPP, Theorem 1.2] to MTCs of dimension congruent to 2 modulo 4 of rank upto 46.

The road map of the paper is as follows. A brief introduction to fusion and modular categories is given in Section 2. In Section 3, we first provide basic results on MTCs of dimension congruent to 2 modulo 4, and later prove Theorem 3.9 by using modular subcategories and equivariantization results. Then, in Section 4, we present some general results useful for the classification of MTCs. In Section 5, we deduce the relationship of solvability between MTCs of dimension congruent to 2 modulo 4 and those with odd dimension. Next, in Section 6, we provide our algorithms for computing the dimensions of particular simple objects in an MTC of dimension not divisible by 4 and in Section 7, we provide our plans for future work.

## 2. Preliminaries

We work over an algebraically closed field $\mathbf{k}$ of characteristic zero. We let Vec be the category of finite dimensional vector spaces over $\mathbf{k}$, and $\mathrm{Rep}(G)$ be the category of finite dimensional representations of a finite group $G$. We refer the reader to [ENO1, EGNO] for the notions of tensor and fusion categories used throughout.

2.1. **Fusion categories.** A *fusion category* $\mathcal{C}$ is a semisimple rigid tensor category over $\mathbf{k}$ with finitely many isomorphism classes of simple objects. We let $\mathbf{1}$ be its identity object, and $\mathcal{O}(\mathcal{C})$ be the set of isomorphism classes of simple objects in $\mathcal{C}$, so that $\mathrm{rank}(\mathcal{C}) = |\mathcal{O}(\mathcal{C})|$.

For an object $X \in \mathcal{C}$, let $X^* \in \mathcal{C}$ denote its dual. We say $X$ is *self-dual* if $X \cong X^*$, and *non-self-dual* if $X \ncong X^*$. We say a category $\mathcal{C}$ is maximally-non-self-dual (MNSD) if $X \ncong X^*$ for all simple $X \ncong \mathbf{1}$ in $\mathcal{C}$.

A *braiding* on a fusion category $\mathcal{C}$ is a natural isomorphism

$$c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X,$$

for all $X, Y \in \mathcal{C}$, satisfying the so-called hexagonal diagrams, see [EGNO, Definition 8.1.1]. We will say that $\mathcal{C}$ is *braided* if it is equipped with a braiding.

2.1.1. *Frobenius-Perron dimension.* Let $\mathcal{C}$ be a fusion category. We denote by $\mathcal{K}(\mathcal{C})$ its Grothendieck ring, see e.g. [EGNO, Section 4.5]. For $X$ in $\mathcal{C}$, we will denote its class $X$ in $\mathcal{K}(\mathcal{C})$ with the same notation. There is a unique ring homomorphism $\mathrm{FPdim} : \mathcal{K}(\mathcal{C}) \to \mathbb{R}$, called *Frobenius-Perron dimension*, such that $\mathrm{FPdim}(X) \geq 1$ for any object $X \neq 0$, see [EGNO, Proposition 3.3.4]. The *Frobenius-Perron dimension* $\mathrm{FPdim}(\mathcal{C})$ of $\mathcal{C}$ is defined by

$$\mathrm{FPdim}(\mathcal{C}) := \sum_{X \in \mathcal{O}(\mathcal{C})} \mathrm{FPdim}(X)^2.$$

If $\mathrm{FPdim}(\mathcal{C})$ is an integer, we have that $\mathcal{C}$ is *weakly integral*. Similarly, if $\mathrm{FPdim}(X)$ is an integer for all simple objects $X$, then $\mathcal{C}$ is *integral*.

2.1.2. *Pointed fusion categories.* Let $\mathcal{C}$ be a fusion category. An object $X$ in $\mathcal{C}$ is said to be *invertible* if its evaluation $X^* \otimes X \to \mathbf{1}$ and coevaluation $\mathbf{1} \to X \otimes X^*$ maps are isomorphisms, see [EGNO, Definition 2.10.1]. Equivalently, an object $X$ is invertible if $\mathrm{FPdim}(X) = 1$.

A fusion category $\mathcal{C}$ is *pointed* if all simple objects in $\mathcal{C}$ are invertible. It is well known that any pointed fusion category $\mathcal{C}$ is equivalent to the category of finite dimensional $G$-graded vector spaces $\mathrm{Vec}_G^{\omega}$, where $G$ is a finite group and $\omega$ is a 3-cocycle on $G$ with coefficients in $\mathbf{k}^{\times}$ codifying the associativity constraint.

We denote the group of isomorphism classes of invertible objects of $\mathcal{C}$ by $\mathcal{G}(\mathcal{C})$. The largest pointed subcategory of $\mathcal{C}$ will be written as $\mathcal{C}_{\mathrm{pt}}$.

2.1.3. *The universal grading.* Let $G$ be a finite group. A *$G$-grading* on a fusion category $\mathcal{C}$ is a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g,$$

such that $\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$, $\mathbf{1} \in \mathcal{C}_e$, and the dualizing functor maps $\mathcal{C}_g$ to $\mathcal{C}_{g^{-1}}$, refer to [EGNO, Section 4.14]. Such grading is said to be *faithful* if $\mathcal{C}_g \neq 0$ for all $g \in G$. It was shown in [ENO1, Proposition 8.20] that for a faithful grading all the components $\mathcal{C}_g$ have the same Frobenius-Perron dimension, and so

$$\mathrm{FPdim}(\mathcal{C}) = |G| \, \mathrm{FPdim}(\mathcal{C}_e).$$

By [GN], any fusion category $\mathcal{C}$ admits a canonical faithful grading $\mathcal{C} = \bigoplus_{g \in U(\mathcal{C})} \mathcal{C}_g$, called the *universal grading*. Its trivial component coincides with the *adjoint subcategory* $\mathcal{C}_{\mathrm{ad}}$ of $\mathcal{C}$, defined as the fusion subcategory generated by $X \otimes X^*$ for all $X \in \mathcal{O}(\mathcal{C})$. If $\mathcal{C}$ is equipped with a braiding, then $\mathcal{U}(\mathcal{C})$ is abelian. Moreover, if $\mathcal{C}$ is modular then $\mathcal{U}(\mathcal{C})$ is isomorphic to $\mathcal{G}(\mathcal{C})$ [GN, Theorem 6.3].

2.1.4. *Solvable fusion categories.* A fusion category $\mathcal{C}$ is *solvable* if it is Morita equivalent to a cyclically nilpotent fusion category, see [ENO2]. The class of solvable fusion categories is closed under the Deligne tensor product, Drinfeld centers, fusion subcategories, and extension and equivariantization by solvable groups, see [ENO2, Propositions 4.1 and 4.5].

2.2. **Modular tensor categories.** Let $\mathcal{C}$ be a braided fusion category. A *pivotal structure* on $\mathcal{C}$ is a natural isomorphism $\psi : \mathrm{Id} \xrightarrow{\sim} (-)^{**}$, i.e., an isomorphism between the double dual and identity functors, see [BW, EGNO]. With a pivotal structure we can define the left and right trace of a morphism $X \to X$, see e.g. [EGNO, Section 4.7]. If, for any such morphism its left trace equals its right trace, then the pivotal structure is called *spherical*. A *premodular* tensor category is a braided fusion category equipped with a spherical structure.

Equivalently, a pre-modular category is a braided fusion category endowed with a compatible ribbon structure. Let $\mathcal{C}$ have braiding $\sigma_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$. A *ribbon structure* on $\mathcal{C}$ is a natural isomorphism $\theta_X : X \xrightarrow{\cong} X$ for all $X \in \mathcal{C}$, satisfying

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ \sigma_{Y,X} \circ \sigma_{X,Y}, \tag{1}$$

and $(\theta_X)^* = \theta_{X^*}$ for all $X, Y \in \mathcal{C}$.

Let $\mathcal{C}$ be a premodular tensor category, with braiding $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$. The *S-matrix* $S$ of $\mathcal{C}$ is defined by $S := (s_{X,Y})_{X,Y\in\mathcal{O}(\mathcal{C})}$, where $s_{X,Y}$ is the trace of $c_{Y,X}c_{X,Y} : X \otimes Y \to X \otimes Y$. If $\mathcal{C}$ is a premodular tensor category, it is said to be *modular* if its $S$-matrix is non-degenerate, i.e. invertible.

In this work, we require the terms "premodular tensor category" and "modular tensor category" to be fusion categories, so they must imply semisimplicity. This is a slight abuse of terminology, which we adopt to be consistent with prior papers. Also, the term "modular category" is equivalent to a "modular tensor category" as we defined in this paper.

2.2.1. *Centralizers.* Let $\mathcal{C}$ be a braided fusion category with braiding $c_{X,Y} : X\otimes Y \xrightarrow{\sim} Y\otimes X$, and let $\mathcal{K}$ be a fusion subcategory of $\mathcal{C}$. The *Müger centralizer* of $\mathcal{K}$ is the fusion subcategory $\mathcal{K}'$ of $\mathcal{C}$ consisting of all objects $Y$ in $\mathcal{C}$ such that

$$c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y}, \quad \text{for all } X \in \mathcal{K}, \tag{2}$$

see [EGNO, Definition 8.20.1].

A symmetric fusion category $\mathcal{C}$ is called *Tannakian* if it is equivalent as a braided fusion category to the category $\mathrm{Rep}(G)$ for some finite group $G$ with braiding given by the usual flip of vector spaces. It is also known that all odd-dimensional symmetric fusion categories are Tannakian [DGNO1, Corollary 2.50].

We say a fusion subcategory $\mathcal{D}$ of $\mathcal{C}$ is a *modular subcategory* if and only if $\mathcal{D}\cap\mathcal{D}' = \mathrm{Vec}$. So, $\mathcal{D}$ is modular if and only if $\mathcal{D}' = \mathrm{Vec}$.

**Remark 2.1.** [CP, Remark 2.2] If $\mathcal{C}$ is a braided fusion category, then $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is symmetric.

2.2.2. *Equivariantization and de-equivariantization.* Let $\mathcal{C}$ be a fusion category with an action of a finite group $G$, see [EGNO, Definition 4.15.1]. Following [EGNO, Section 2.7], one can construct a fusion category $\mathcal{C}^G$ of $G$-equivariant objects in $\mathcal{C}$, called the *equivariantization* of $\mathcal{C}$ by $G$. If the action of $G$ on $\mathcal{C}$ is braided, see [EGNO, Definition 8.23.6], then $\mathcal{C}^G$ is also braided. Moreover, $\mathrm{Rep}(G)$ is identified with a Tannakian subcategory of $\mathcal{C}^G$ via the canonical embedding $\mathrm{Rep}(G) \hookrightarrow \mathcal{C}^G$.

On the other hand, suppose that $\mathcal{C}$ is a braided fusion category with a Tannakian subcategory $\mathrm{Rep}(G)$. Then, the *de-equivariantization* $\mathcal{C}_G$ of $\mathcal{C}$ with respect to $\mathrm{Rep}(G)$ is an important category rooting from $\mathrm{Rep}(G)$, see [EGNO, Theorem 8.23.3] for the construction. The category $\mathcal{C}_G$ is a braided fusion category. Also, we have an equivalence of braided fusion categories $\mathcal{C} \cong (\mathcal{C}_G)^G$.

Note that dimensions are well-behaved under equivariantization and de-equivariantization, see [DGNO1, Proposition 4.26]. In fact, we have $\mathrm{FPdim}(\mathcal{C}^G) = |G|\cdot\mathrm{FPdim}(\mathcal{C})$ and $\mathrm{FPdim}(\mathcal{C}_G) = \frac{1}{|G|}\cdot\mathrm{FPdim}(\mathcal{C})$.

Let $\mathcal{C}$ be an MTC such that $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is odd-dimensional. By [CP, Remark 2.2], $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is symmetric and thus Tannakian [DGNO1, Corollary 2.50]. So, $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}} \cong \mathrm{Rep}(G)$ for some finite group $G$, and by [ENO2, Remark 2.3], the de-equivariantization $(\mathcal{C}_{\mathrm{ad}})_G$ is modular.

## 3. MTCs of dimension congruent to 2 modulo 4

In this section, we prove the main result of our paper, namely that all MTCs of dimension congruent to 2 modulo 4 decompose as $\mathcal{D}\boxtimes\mathrm{Vec}^w_{\mathbb{Z}_2}$ where $w$ is a 3-cocycle on $\mathbb{Z}_2$ and $\mathcal{D}$ is an odd-dimensional MTC. We first prove that this holds for when $|\mathcal{G}(\mathcal{C})|$ is even, and later use this result as well as properties of equivariantization to prove that $|\mathcal{G}(\mathcal{C})|$ cannot be odd. As a corollary of this result, we deduce that all MTCs of dimension congruent to 2 modulo 4 of rank up to 46 are pointed.

**Lemma 3.1.** *Let $\mathcal{C}$ be an MTC with* $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$. *Then, exactly one of the following hold.*

*(1)* $|\mathcal{G}(\mathcal{C})| \equiv 2 \pmod 4$ *and* $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$ *is odd, or*
*(2)* $|\mathcal{G}(\mathcal{C})|$ *is odd and* $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) \equiv 2 \pmod 4$.

*Proof.* The statement follows from taking equivalence modulo 4 on the equality $\mathrm{FPdim}(\mathcal{C}) = |\mathcal{G}(\mathcal{C})|\,\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$. □

**Corollary 3.2.** *Let $\mathcal{C}$ be an MTC with* $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$. *Then $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ has odd order.*

*Proof.* From Lemma 3.1, $|\mathcal{G}(\mathcal{C})|$ is either odd or congruent to 2 modulo 4. If the former is true, the statement follows trivially. In the latter case, we have that $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$ must be odd from Lemma 3.1. So, since $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$ divides $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$, we have that $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$ is odd. □

**Lemma 3.3.** *Let $\mathcal{C}$ be an MTC with $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$ such that $\mathcal{G}(\mathcal{C}) \cong \mathbb{Z}_2$. Then $\mathcal{C}_{\mathrm{ad}}$ is an odd-dimensional MTC and we have a decomposition $\mathcal{C} \cong \mathcal{C}_{\mathrm{ad}} \boxtimes \mathrm{Vec}_{\mathbb{Z}_2}^{\omega}$ where $\omega$ is a 3-cocycle on $\mathbb{Z}_2$.*

*Proof.* By Corollary 3.2, $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ is an odd order subgroup of $\mathcal{G}(\mathcal{C}) \cong \mathbb{Z}_2$, and so it must be trivial. By the proof of [CP, Lemma 5.2], this implies that $\mathcal{C}_{\mathrm{ad}}$ and $\mathcal{C}_{\mathrm{pt}}$ are modular, and we have the factorization $\mathcal{C} \cong \mathcal{C}_{\mathrm{ad}} \boxtimes \mathcal{C}_{\mathrm{pt}} \cong \mathcal{C}_{\mathrm{ad}} \boxtimes \mathrm{Vec}_{\mathbb{Z}_2}^{\omega}$ where $\omega$ is a 3-cocycle on $\mathbb{Z}_2$. Lastly, by Lemma 3.1, $\mathcal{C}_{\mathrm{ad}}$ is odd-dimensional. □

**Remark 3.4.** Let $\mathcal{C}$ be an MTC of dimension congruent to 2 modulo 4. If $|\mathcal{G}(\mathcal{C})|$ is even, then notice from Lemma 3.1 that $|\mathcal{G}(\mathcal{C})| \equiv 2 \pmod 4$. So, we must have that $\mathcal{G}(\mathcal{C}) \cong \mathbb{Z}_2 \times G$ for an odd-order abelian group $G$, which implies that $\mathcal{G}(\mathcal{C})$ has exactly one invertible of order 2.

**Lemma 3.5.** *Let $\mathcal{C}$ be an MTC with $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$. Suppose that $|\mathcal{G}(\mathcal{C})|$ is even and let $f$ be the invertible of order 2 in $\mathcal{C}$. Then, exactly one of the following hold.*

- *The fusion subcategory $\mathcal{C}[f]$ is modular, or*
- *The fusion subcategory $\mathcal{C}[f]$ is equivalent to* sVec *and $f$ is a fermion.*

*Proof.* Consider the fusion subcategory $\mathcal{C}[f]$. If $\mathcal{C}[f] \cap \mathcal{C}[f]' = \mathrm{Vec}$, then $\mathcal{C}[f]$ be modular and the claim holds in this case.

On the other hand, suppose $\mathcal{C}[f] \cap \mathcal{C}[f]'$ is non-trivial. Then, since the order of $f$ is 2, we have that $\mathcal{C}[f] \cap \mathcal{C}[f]' = \mathcal{C}[f]$. Thus, $\mathcal{C}[f]$ is a symmetric subcategory of $\mathcal{C}$. Suppose first that $\mathcal{C}[f]$ is Tannakian and is equivalent to $\mathrm{Rep}(\mathbb{Z}_2)$. Then, from [N2, Section 4], we would have that $|G|^2 = 4$ divides $\mathrm{FPdim}(\mathcal{C})$, which is not possible. So, $\mathcal{C}[f]$ must be equivalent to sVec, and we then have that $f$ is a fermion [BGHNPRW, Definition 2.1]. So, the lemma is proven. □

Recall that if $|\mathcal{G}(\mathcal{C})| = 2$, then $\mathcal{C}[f]$ is modular, see Lemma 3.3. This is also the case in the more general setting when the group of invertibles has even order. In the following proposition, we show that when the group of invertibles has even order, the category can always be determined in terms of an odd-dimensional modular category.

**Proposition 3.6.** *Let $\mathcal{C}$ be an MTC with $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$. If $|\mathcal{G}(\mathcal{C})|$ is even, then the pointed subcategory of rank 2 is a modular subcategory of $\mathcal{C}$. In this case, we have that $\mathcal{C} \cong \widetilde{\mathcal{C}} \boxtimes \mathrm{semion}$, where $\widetilde{\mathcal{C}}$ is an odd-dimensional modular category and the* semion *is a rank 2 pointed modular category.*

*Proof.* Consider first when $|\mathcal{G}(\mathcal{C})| = 2$. Then, by Lemma 3.3, we have that $\mathcal{C} \cong \mathcal{C}_{\text{ad}} \boxtimes \mathcal{C}_{\text{pt}}$, with $\mathcal{C}_{\text{pt}}$ a rank 2 pointed modular category, that is, $\mathcal{C}_{\text{pt}} \cong$ semion. Moreover, $\mathcal{C}_{\text{ad}}$ is an odd-dimensional modular category in this case.

Now, consider when $|\mathcal{G}(\mathcal{C})| > 2$. From Lemma 3.5, since $|\mathcal{G}(\mathcal{C})| > 2$, there is an invertible $f$ of order 2 and $\mathcal{C}[f]$ is either modular or equivalent to sVec. If it is modular, then the claim holds so suppose the latter.

From [BGHNPRW, Proposition 2.4], $f$ is a fermion, and there is a faithful $\mathbb{Z}_2$-grading of $\mathcal{C} \cong \mathcal{C}_0 \oplus \mathcal{C}_1$ such that a simple object $X \in \mathcal{C}_0$ if $\epsilon_X = 1$ and $X \in \mathcal{C}_1$ if $\epsilon_X = -1$. For the definition of $\epsilon_X$, see [BGHNPRW, Proposition 2.3(ii)].

Suppose first that $\mathcal{C}_1$ contains at least one invertible object $h$. In a similar argument as in [CGP, Proposition 3.17], we have that $\mathcal{C}_0$ and $\mathcal{C}_1$ contain the same number of invertibles. Let the order of $h \in \mathcal{C}_1$ be $n$. Then, we have that $h^n \in \mathcal{C}_0$. Notice that this implies that $n$ must be even, which follows a similar argument than the one in the proof of [N1, Theorem 4.1(i)]. Therefore, the order of all invertibles in $\mathcal{C}_1$ must be even. Since $\mathcal{C}_0$ and $\mathcal{C}_1$ have the same number of invertible objects, all invertible objects of even order should be in $\mathcal{C}_1$. However, from [BGHNPRW, Proposition 2.3(ii)], we have that $f \in \mathcal{C}_0$, and so, this is impossible.

We then have that all invertible objects are in $\mathcal{C}_0$, and that $\epsilon_h = 1$ for all invertible objects $h$ in $\mathcal{C}$. So, from [BGHNPRW, Proposition 2.3(ii)], we have that

$$s_{f,h} = \epsilon_h \operatorname{FPdim}(h) = 1$$

for all invertibles $h \in \mathcal{C}$.

Thus, from [M, Proposition 2.5], $f \in (\mathcal{C}_{\text{pt}})' \cong \mathcal{C}_{\text{ad}}$ [GN, Corollary 6.9]. However, recall from Lemma 3.2 that $|\mathcal{G}(\mathcal{C}_{\text{ad}})|$ has odd order. Thus, a contradiction is derived in this case and the theorem is proven. □

We will now use this proposition to prove a result about the parity of the number of invertibles in $(\mathcal{C}_{\text{ad}})_{\mathcal{G}(\mathcal{C}_{\text{ad}})}$.

**Proposition 3.7.** *Let $\mathcal{C}$ be an MTC with $\operatorname{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$, and suppose that $(\mathcal{C}_{\text{ad}})_{\text{pt}} \cong \operatorname{Rep}(G)$ for a non-trivial odd-order group $G$. Then, $(\mathcal{C}_{\text{ad}})_G$ contains an odd number of invertibles.*

*Proof.* If $\mathcal{C}_{\text{ad}}$ is odd-dimensional, the claim immediately follows. So, suppose that $\mathcal{C}_{\text{ad}}$ has dimension congruent to 2 modulo 4. Then, $(\mathcal{C}_{\text{ad}})_G$ is an MTC of dimension congruent to 2 modulo 4, as well. For the sake of contradiction, suppose there are an even number of invertibles in $\mathcal{D} := (\mathcal{C}_{\text{ad}})_G$. In particular, we have that $|\mathcal{G}(\mathcal{D})| \equiv 2 \pmod 4$. Thus, $\mathcal{D}$ has exactly one invertible $h$ of order 2. Moreover, it follows from Theorem 3.6 that $\mathcal{D} \cong \operatorname{Vec}_{\mathbb{Z}_2}^{\omega} \boxtimes \widetilde{D}$ where $w$ is a 3-cocycle on $\mathbb{Z}_2$ and $\widetilde{D}$ is an odd-dimensional MTC. Notice that the action of $G$ on $\operatorname{Vec}_{\mathbb{Z}_2}^{\omega}$ is trivial on the objects (and morphisms) since the order of $G$ is odd.

Now, we can consider the exact sequence given in [BN, Remark 3.1] that describes the invertibles in the $G$-equivariantization. Then, the exact sequence in this case is given by

$$1 \to \widehat{G} \to \mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \to G_0(\mathcal{D}) \to 1,$$

where $\hat{G}$ and $G_0(\mathcal{D})$ are defined as in [BN, Remark 3.1]. Furthermore, since $G \cong \mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ is abelian and the sequence above is exact, it follows that $G_0(\mathcal{D})$ should be trivial, which contradicts that the action of $G$ on the semion is trivial. $\square$

Finally, to show that $|\mathcal{G}(\mathcal{C})|$ must be even for $\mathcal{C}$ an MTC of dimension congruent to 2 modulo 4, we can use an induction argument.

**Proposition 3.8.** *Let $\mathcal{C}$ be an MTC with $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$. Then $|\mathcal{G}(\mathcal{C})|$ is even.*

*Proof.* Suppose, for the sake of contradiction, that there is an MTC with an odd number of invertibles, and let $\mathcal{C}$ be such an MTC of the smallest dimension. Notice that $\mathcal{C}$ must have dimension greater than 2 because if not, then $\mathcal{C}$ would be pointed, which would imply that $|\mathcal{G}(\mathcal{C})|$ is even. So, we must have that $\mathrm{FPdim}(\mathcal{C}) \geq 6$.

If $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 1$, then recall by the proof of [CP, Lemma 5.2] that $\mathcal{C}_{\mathrm{ad}}$ is an MTC and we have a factorization $\mathcal{C} \cong \mathcal{C}_{\mathrm{ad}} \boxtimes \mathcal{C}_{\mathrm{pt}}$. By assumption, $\mathcal{C}_{\mathrm{pt}}$ is odd-dimensional, and so $\mathcal{C}_{\mathrm{ad}}$ is an MTC of dimension congruent to 2 modulo 4. This contradicts that $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 1$, since MTCs of dimension congruent to 2 modulo 4 have at least one non-trivial invertible object, see [BP, Corollary 10.11].

Hence, we should have that $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| > 1$. Consider the de-equivariantization of $\mathcal{C}_{\mathrm{ad}}$ by $G := \mathcal{G}(\mathcal{C}_{\mathrm{ad}})$. Since $|\mathcal{G}(\mathcal{C})|$ is odd, we must have that $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) \equiv 2 \pmod 4$ from Lemma 3.1. Notice that $(\mathcal{C}_{\mathrm{ad}})_G$ is an MTC with $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_G) \equiv 2 \pmod 4$, and since $G$ is not trivial, we get that $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_G) < \mathrm{FPdim}(\mathcal{C})$. Again, since by assumption $\mathcal{C}$ is an MTC of the smallest dimension with $|\mathcal{G}(\mathcal{C})|$ odd, $(\mathcal{C}_{\mathrm{ad}})_G$ must have an even number of invertibles, but this is not possible from Proposition 3.7.

So, neither cases are possible, and the proposition is proven. $\square$

From the previous results, we immediately obtain a classification of all MTCs of dimension congruent to 2 modulo 4 in terms of the odd-dimensional ones.

**Theorem 3.9.** *Let $\mathcal{C}$ be an MTC with $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$. Then, $\mathcal{C} \cong \widetilde{\mathcal{C}} \boxtimes \mathrm{Vec}_{\mathbb{Z}_2}^{w}$, where $\widetilde{\mathcal{C}}$ is an odd-dimensional MTC and $w$ is a 3-cocycle on $\mathbb{Z}_2$.*

*Proof.* From Proposition 3.8, we must have that $|\mathcal{G}(\mathcal{C})|$ is even, and the claim follows directly from Proposition 3.6. $\square$

**Corollary 3.10.** *Let $\mathcal{C}$ be an MTC of dimension congruent to 2 modulo 4. If $\mathrm{rank}(\mathcal{C}) \leq 46$, then $\mathcal{C}$ is pointed.*

*Proof.* Since $\mathcal{C}$ has dimension congruent to 2 modulo 4, it factorizes as $\mathcal{D} \boxtimes \mathrm{Vec}_{\mathbb{Z}_2}$ for $\mathcal{D}$ an odd-dimensional MTC and $w$ a 3-cocycle on $\mathbb{Z}_2$. Since $\mathrm{rank}(\mathcal{D}) \leq 23$, it must be pointed [BR, CGP, CP], and so, the claim follows. □

## 4. General results for classification

In this section, we present several results for MTCs that could be used for the classification of odd-dimensional MTCs, which would lead to the classification of MTCs of dimension congruent to 2 modulo 4, see Section 3. For the rest of this section, for an MTC $\mathcal{C}$ consider its universal grading $\mathcal{C} = \bigoplus_{g \in \mathcal{U}(\mathcal{C})} \mathcal{C}_g$, see Section 2.1.3.

The following proof resembles that of [CP, Proposition 5.6].

**Lemma 4.1.** *Let $\mathcal{C}$ be an MTC such that $\mathcal{C}_{\mathrm{pt}} \subseteq \mathcal{C}_{\mathrm{ad}}$ and $|\mathcal{G}(\mathcal{C})|$ is odd. Let $p$ be the smallest prime divisor of $\mathcal{G}(\mathcal{C})$. Then, $\mathrm{rank}(\mathcal{C}_g) \geq p$ for all non-trivial $g$.*

*Proof.* Let $X$ be a simple object in $\mathcal{C}$ fixed by the action of $\mathcal{G}(\mathcal{C})$. Since $\mathcal{C}_{\mathrm{pt}} \subseteq \mathcal{C}_{\mathrm{ad}}$ and $(\mathcal{C}_{\mathrm{ad}})' \cong \mathcal{C}_{\mathrm{pt}}$ [GN, Corollary 6.9], we have that $\mathcal{C}_{\mathrm{pt}}$ is symmetric, and so, since $|\mathcal{G}(\mathcal{C})|$ is odd, from [DGNO2, Corollary 2.7] we have that $\theta_g = 1$ for all $g \in \mathcal{G}(\mathcal{C})$. By the balancing equation, see [EGNO, Proposition 8.13.8], we get

$$s_{g,X} = \theta_g^{-1}\theta_X^{-1}\theta_X \,\mathrm{FPdim}(X) = \mathrm{FPdim}(X) = \mathrm{FPdim}(X)\,\mathrm{FPdim}(g)$$

for all invertibles $g$. It follows from [M, Proposition 2.5] that $X \in \mathcal{C}_{\mathrm{ad}}$. Now, consider a non-trivial component $\mathcal{C}_g$. Since no simple objects are fixed by the action of left multiplication by $\mathcal{G}(\mathcal{C})$ on $\mathcal{C}_g$, there must be an orbit of size at least $p$, the smallest prime divisor of $|\mathcal{G}(\mathcal{C})|$. Thus, $\mathrm{rank}(\mathcal{C}_g) \geq p$, and the lemma is proven. □

**Corollary 4.2.** *Let $\mathcal{C}$ be an MTC such that $\mathcal{C}_{\mathrm{pt}} \subseteq \mathcal{C}_{\mathrm{ad}}$ and $|\mathcal{G}(\mathcal{C})| > 1$ is odd. Let $p$ be the smallest prime divisor of $\mathcal{G}(\mathcal{C})$. Then, $p \leq \frac{\mathrm{rank}(\mathcal{C})}{|\mathcal{G}(\mathcal{C})|}$.*

*Proof.* From Lemma 4.1, $\mathrm{rank}(\mathcal{C}_g) \geq p$ for all non-trivial $g$. Also, since $\mathcal{C}_{\mathrm{pt}} \subseteq \mathcal{C}_{\mathrm{ad}}$, we have that $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) \geq p$, as well, and the result follows. □

**Remark 4.3.** Let $\mathcal{C}$ be an MTC. If $\mathcal{C}_g \subseteq \mathcal{C}_{\mathrm{pt}}$ for all $g \neq 1$, then $\mathcal{C}$ is pointed.

*Proof.* Since there are $|\mathcal{G}(\mathcal{C})|$ non-empty components and the unit is in $\mathcal{C}_{\mathrm{ad}}$, all non-trivial components must contain exactly one invertible object. Thus, all components have dimension 1, and $\mathcal{C}$ must be pointed. □

### 4.1. **Integral MTCs.**

In this subsection, we present results for integral MTCs. In particular, we focus on properties regarding the size of the group of invertibles.

**Lemma 4.4.** *Let $\mathcal{C}$ be an integral non-pointed MTC. Then,*

$$|\mathcal{G}(\mathcal{C})| \leq \left\lceil \frac{2\,\mathrm{rank}(\mathcal{C})}{3} \right\rceil.$$

*Proof.* Assume, for the sake of contradiction, that $|\mathcal{G}(\mathcal{C})| > \left\lceil \frac{2\operatorname{rank}(\mathcal{C})}{3} \right\rceil$. Let $g_1, g_2, \ldots, g_l$ be all the elements of $\mathcal{G}(\mathcal{C})$ such that $\mathcal{C}_{g_1}, \ldots, \mathcal{C}_{g_l}$ have exactly 1 simple object. Notice that these objects must be non-invertible; otherwise $\operatorname{FPdim}(\mathcal{C}_g) = 1$ for all $g \in \mathcal{G}(\mathcal{C})$, contradicting that $\mathcal{C}$ is not pointed. Thus, $l \leq \operatorname{rank}(\mathcal{C}) - |\mathcal{G}(\mathcal{C})|$.

We will now show that there is another component that has exactly one simple object, leading to a contradiction. Since $\mathcal{C}$ is not pointed, $|\mathcal{G}(\mathcal{C})| < \operatorname{rank}(\mathcal{C})$. Also, since $|\mathcal{G}(\mathcal{C})| > \left\lceil \frac{2\operatorname{rank}(\mathcal{C})}{3} \right\rceil$, we have

$$2\operatorname{rank}(\mathcal{C}) < 3|\mathcal{G}(\mathcal{C})|.$$

It follows that

$$\operatorname{rank}(\mathcal{C}) < 3|\mathcal{G}(\mathcal{C})| - \operatorname{rank}(\mathcal{C}) = 2|\mathcal{G}(\mathcal{C})| - (\operatorname{rank}(\mathcal{C}) - |\mathcal{G}(\mathcal{C})|).$$

So, for all integers $k$ in $[0, \operatorname{rank}(\mathcal{C}) - |\mathcal{G}(\mathcal{C})|]$, we obtain that

$$|\mathcal{G}(\mathcal{C})| - k < \operatorname{rank}(\mathcal{C}) - k < 2|\mathcal{G}(\mathcal{C})| - 2k.$$

Since by assumption $\mathcal{G}(\mathcal{C}) > \frac{\operatorname{rank}(\mathcal{C})}{2}$, we have that $|\mathcal{G}(\mathcal{C})| - k \geq 2|\mathcal{G}(\mathcal{C})| - \operatorname{rank}(\mathcal{C})$ must be positive. So, we obtain

$$1 < \frac{\operatorname{rank}(\mathcal{C}) - k}{|\mathcal{G}(\mathcal{C})| - k} < 2. \tag{3}$$

Since $l$ is between 0 and $\operatorname{rank}(\mathcal{C}) - |\mathcal{G}(\mathcal{C})|$ (inclusive), we obtain that

$$1 < \frac{\operatorname{rank}(\mathcal{C}) - l}{|\mathcal{G}(\mathcal{C})| - l} < 2.$$

If each of the remaining components had at least two objects, then $\frac{\operatorname{rank}(\mathcal{C}) - l}{|\mathcal{G}(\mathcal{C})| - l}$ would be greater than or equal to 2. So, at least one remaining component must have exactly one simple object, contradicting that $\mathcal{C}_{g_1}, \ldots \mathcal{C}_{g_l}$ were the only such components. □

**Lemma 4.5.** *Let $\mathcal{C}$ be a non-pointed integral MTC. Then every odd prime $p$ dividing $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$ satisfies*

$$p \leq \frac{\operatorname{rank}(\mathcal{C}) - |\mathcal{G}(\mathcal{C})|}{2}.$$

*Proof.* From [CP, Lemma 5.1] it follows that that

$$\operatorname{rank}(\mathcal{C}) \geq \operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) + |\mathcal{G}(\mathcal{C})| + p - 2 \geq 2p + |\mathcal{G}(\mathcal{C})| - 2,$$

and so

$$p \leq \frac{\operatorname{rank}(\mathcal{C}) - |\mathcal{G}(\mathcal{C})|}{2} + 1. \tag{4}$$

Note that the equality in the equation above is only possible when $\operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) = p$, that is, when $\mathcal{C}_{\mathrm{ad}}$ is pointed. But in such case $\mathcal{C}$ is nilpotent, and so from [GN, Corollary 5.3] we know that $\operatorname{FPdim}(X)^2$ divides $\operatorname{FPdim}(\mathcal{C}_{\mathrm{ad}}) = p$ for all simple objects $X$ in $\mathcal{C}$. This then implies that $\mathrm{FPdim}(X) = 1$ for all simple $X \in \mathcal{C}$, which is a contradiction since $\mathcal{C}$ is non-pointed. Hence, the inequality in Equation 4 is strict and the result follows. □

**Lemma 4.6.** *Let $\mathcal{C}$ be an integral MTC such that $\mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \cong \mathbb{Z}_p$ for a prime $p$. Then,*

$$\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_p}) \geq \frac{9\,\mathrm{rank}(\mathcal{C}) - 8|\mathcal{G}(\mathcal{C})|}{p|\mathcal{G}(\mathcal{C})|}.$$

*Proof.* Since all non-invertible objects in $\mathcal{C}$ have dimension at least 3,

$$\mathrm{FPdim}(\mathcal{C}) \geq |\mathcal{G}(\mathcal{C})| + 9(\mathrm{rank}(\mathcal{C}) - |\mathcal{G}(\mathcal{C})|) = 9\,\mathrm{rank}(\mathcal{C}) - 8|\mathcal{G}(\mathcal{C})|.$$

The lemma follows since $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_p}) = \frac{\mathrm{FPdim}(\mathcal{C})}{p|\,\mathcal{G}(\mathcal{C})\,|}$. □

**Remark 4.7.** Let $\mathcal{C}$ be an integral non-pointed MTC and suppose that $\mathrm{rank}(\mathcal{C}_g) = \mathrm{FPdim}(\mathcal{C}_g)$ for some $g \in \mathcal{G}(\mathcal{C})$. Then, $\mathrm{rank}(\mathcal{C}_g) \geq 9$.

*Proof.* Since $\mathcal{C}$ is non-pointed, it must have at least one non-invertible object $X \in \mathcal{C}_h$. Hence $\mathrm{rank}(\mathcal{C}_g) = \mathrm{FPdim}(\mathcal{C}_g) = \mathrm{FPdim}(\mathcal{C}_h) \geq \mathrm{FPdim}(X)^2 = 9$, as desired. □

**Lemma 4.8.** *Let $\mathcal{C}$ be an integral MTC. Suppose that $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = p$ for an odd prime $p$, and that there exists a non-invertible simple object $X \in \mathcal{C}$ such that $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = l\,\mathrm{FPdim}(X)^2$ for some integer $l$ not divisible by $p$. Then, at least $p$ non-invertible simple objects in $\mathcal{C}_{\mathrm{ad}}$ are not fixed by the action of left multiplication by $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$.*

*Proof.* Since $\mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \cong \mathbb{Z}_p$, the orbits of its action by left multiplication on $\mathcal{C}_{\mathrm{ad}}$ are either trivial or of size $p$. Assume, for the sake of contradiction, that the orbits of all non-invertible simple objects are trivial. That is, all non-invertible simple objects are fixed by the action, and thus have Frobenius-Perron dimension divisible by $p$. Then,

$$l\,\mathrm{FPdim}(X)^2 = \mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) \equiv p \pmod{p^2},$$

and thus there is a non-negative integer $m$ for which $l\,\mathrm{FPdim}(X)^2 = p^2 m + p = p(pm + 1)$. Since $l$ is not divisible by $p$, $\mathrm{FPdim}(X)^2$ must be divisible by $p^2$, which leads to a contradiction in the congruence above. □

For a simple object $X$ in $\mathcal{C}$, let $m_X$ be the integer such that

$$\mathrm{FPdim}(\mathcal{C}) = m_X\,\mathrm{FPdim}(X)^2. \tag{5}$$

**Lemma 4.9.** *Let $\mathcal{C}$ be an integral MTC such that $|\mathcal{G}(\mathcal{C})| = |\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = p$ for an odd prime $p$. Then there exists a non-invertible simple object $X \in \mathcal{C}$ such that $\mathrm{FPdim}(X)^2$ divides $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$.*

*Proof.* Suppose, for the sake of contradiction, that for all $X \in \mathcal{O}(\mathcal{C})$, $\mathrm{FPdim}(X)^2$ does not divide $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$. Let $X \in \mathcal{O}(\mathcal{C})$ and let $m_X$ be as in Equation (5). Then $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = \frac{m_X\,\mathrm{FPdim}(X)^2}{|\mathcal{G}(\mathcal{C})|}$ and since $\mathrm{FPdim}(X)$ does not divide $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$, we must have that $p =$

$|\mathcal{G}(\mathcal{C})|$ divides $\mathrm{FPdim}(X)$. Since this is true for all $X \in \mathcal{O}(\mathcal{C})$, we have that

$$\mathrm{FPdim}(\mathcal{C}) = |\mathcal{G}(\mathcal{C})| + \sum_{\substack{X \in \mathcal{O}(\mathcal{C}) \\ X \notin \mathcal{G}(\mathcal{C})}} \mathrm{FPdim}(X)^2 \equiv p \pmod{p^2}.$$

But $\mathrm{FPdim}(X)^2$ divides $\mathrm{FPdim}(\mathcal{C})$, so $\mathrm{FPdim}(\mathcal{C}) \equiv 0 \pmod{p^2}$, a contradiction. □

**Corollary 4.10.** *Let $\mathcal{C}$ be an integral non-pointed MTC such that $|\mathcal{G}(\mathcal{C})| = |\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = p$ for an odd prime $p$. Then, $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$ is not square-free.*

*Proof.* This is immediate from Lemma 4.9. □

**Lemma 4.11.** *Let $\mathcal{C}$ be an integral MTC. Suppose that $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = p$ for an odd prime $p$ and that all non-invertible objects in $\mathcal{C}_{\mathrm{ad}}$ are fixed by the action of left multiplication by $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$. Then,*

$$p|\mathcal{G}(\mathcal{C})| \equiv m_X \, \mathrm{FPdim}(X)^2 \pmod{p^2|\mathcal{G}(\mathcal{C})|}$$

*for all simple objects $X$ in $\mathcal{C}$, where $m_X$ is as in Equation (5).*

*Proof.* Since all non-invertible objects in $\mathcal{C}_{\mathrm{ad}}$ are fixed by the action of left multiplication by $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$, they have Frobenius-Perron dimension divisible by $p$. So $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = p + p^2 n$ for an integer $n \geq 0$, and thus $\mathrm{FPdim}(\mathcal{C}) = |\mathcal{G}(\mathcal{C})|(p + p^2 n)$. Consider a simple object $X$ in $\mathcal{C}$. Then

$$m_X \, \mathrm{FPdim}(X)^2 = \mathrm{FPdim}(\mathcal{C}) = |\mathcal{G}(\mathcal{C})|(p + p^2 n)$$

and so

$$m_X \, \mathrm{FPdim}(X)^2 \equiv |\mathcal{G}(\mathcal{C})| p \pmod{p^2|\mathcal{G}(\mathcal{C})|},$$

as desired. □

4.2. **Odd-dimensional MTCs.** This subsection focuses on odd-dimensional MTCs, as the classification of these will lead to the classification of MTCs with dimension congruent to 2 modulo 4.

**Lemma 4.12.** *Let $\mathcal{C}$ be an odd-dimensional MTC. Then,*

$$\mathrm{rank}(\mathcal{C}) \equiv |\mathcal{G}(\mathcal{C})| \, \mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod{16}.$$

*Proof.* Consider a non-trivial component $\mathcal{C}_g$. From [CP, Remark 5.5], $\mathrm{rank}(\mathcal{C}_g) \equiv \mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod{16}$ or $\mathrm{rank}(\mathcal{C}_g) \equiv \mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) + 8 \pmod{16}$. In either case,

$$\mathrm{rank}(\mathcal{C}_g) + \mathrm{rank}(\mathcal{C}_{g^{-1}}) \equiv 2 \, \mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod{16},$$

and so

$$\begin{aligned} \mathrm{rank}(\mathcal{C}) &= \mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) + \sum_{g \in \mathcal{G}(\mathcal{C}), g \neq 1} \mathrm{rank}(\mathcal{C}_g) \\ &\equiv \mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) + (|\,\mathcal{G}(\mathcal{C})\,| - 1) \, \mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod{16} \\ &\equiv |\mathcal{G}(\mathcal{C})| \, \mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod{16}, \end{aligned}$$

as desired. □

In the following lemmas, let $0 \leq m_{\mathcal{C}} \leq 7$ such that

$$m_{\mathcal{C}} \equiv |\mathcal{G}(\mathcal{C})| \operatorname{rank}(\mathcal{C}) \pmod 8. \tag{6}$$

**Lemma 4.13.** *Let $\mathcal{C}$ be an odd-dimensional MTC. Then, $m_{\mathcal{C}} \equiv \operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod 8$.*

*Proof.* From Lemma 4.12, since

$$\operatorname{rank}(\mathcal{C}) \equiv |\mathcal{G}(\mathcal{C})| \operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod 8,$$

it follows that

$$\operatorname{rank}(\mathcal{C})|\mathcal{G}(\mathcal{C})| \equiv |\mathcal{G}(\mathcal{C})|^2 \operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod 8.$$

So, since $|\mathcal{G}(\mathcal{C})|$ is odd and all odd squares are congruent to 1 modulo 8, we conclude

$$m_{\mathcal{C}} \equiv \operatorname{rank}(\mathcal{C})|\mathcal{G}(\mathcal{C})| \equiv \operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod 8,$$

as desired. □

**Lemma 4.14.** *Let $\mathcal{C}$ be an odd-dimensional MTC and let $p$ be an odd prime dividing $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$. Then,*

$$|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| \leq \operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) \leq \operatorname{rank}(\mathcal{C}) + m_{\mathcal{C}}(3 - |\mathcal{G}(\mathcal{C})|) - 2p.$$

*Proof.* Recall from [CP, Lemma 5.1(a)] that there exists a non-trivial element $h$ in $\mathcal{U}(\mathcal{C})$ for which $\mathcal{C}_h$ has at least $p$ non-invertible simple objects of the same dimension. Since $\mathcal{G}(\mathcal{C})$ is of odd order, we have that $(\mathcal{C}_h)^* \cong \mathcal{C}_{h^{-1}} \not\cong \mathcal{C}_h$ also has at least $p$ non-invertible simple objects of the same dimension. On the other hand, from Lemma 4.13 we get that $m_{\mathcal{C}} \equiv \operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) \pmod 8$, and so [CP, Remark 5.5] implies that $\operatorname{rank}(\mathcal{C}_g) \geq m_{\mathcal{C}}$ for all $g$ in $\mathcal{U}(\mathcal{C})$. Thus,

$$\operatorname{rank}(\mathcal{C}) \geq \operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) + m_{\mathcal{C}}(|\mathcal{G}(\mathcal{C})| - 3) + 2p,$$

and the desired inequality immediately follows.

□

**Lemma 4.15.** *Let $\mathcal{C}$ be a non-pointed odd-dimensional MTC. If $|\mathcal{G}(\mathcal{C})| = p^k$ for an odd prime $p \geq \frac{\operatorname{rank}(\mathcal{C})+3m_{\mathcal{C}}}{3+m_{\mathcal{C}}}$ and $k \geq 1$, then $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 1$.*

*Proof.* Since $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ is a subgroup of $\mathcal{G}(\mathcal{C})$, it is either trivial or $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = p^i$ for $1 \leq i \leq k$. Suppose, for the sake of contradiction, that the latter is true. Then, from Lemma 4.14 we get that

$$p \leq \operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) \leq \operatorname{rank}(\mathcal{C}) - 2p + 3m_{\mathcal{C}} - m_{\mathcal{C}}p^k \leq \operatorname{rank}(\mathcal{C}) - p(2 + m_{\mathcal{C}}) + 3m_{\mathcal{C}}.$$

It follows that $p \leq \frac{\operatorname{rank}(\mathcal{C})+3m_{\mathcal{C}}}{3+m_{\mathcal{C}}}$. Since by assumption $p \geq \frac{\operatorname{rank}(\mathcal{C})+3m_{\mathcal{C}}}{3+m_{\mathcal{C}}}$, we must have $p = \frac{\operatorname{rank}(\mathcal{C})+3m_{\mathcal{C}}}{3+m_{\mathcal{C}}}$. Then by the inequality above we get that $\operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) = p$. That is, $\mathcal{C}_{\mathrm{ad}}$ is pointed and thus $\mathcal{C}$ is nilpotent. From [GN, Corollary 5.3], $\operatorname{FPdim}(X)^2$ divides $\operatorname{FPdim}(\mathcal{C}_{\mathrm{ad}}) = p$ for all $X \in \mathcal{O}(\mathcal{C})$. This implies that $\text{FPdim}(X) = 1$ for all $X \in \mathcal{O}(\mathcal{C})$, which is a contradiction. Thus, $\mathcal{G}(\mathcal{C}_{\text{ad}}) = 1$, as desired. □

4.3. **Modular Subcategories.** In this subsection, we show the existence of modular subcategories for certain types of MTCs.

**Theorem 4.16.** *Let $\mathcal{C}$ be an MTC of dimension not divisible by 4. Suppose that $|\mathcal{G}(\mathcal{C})|$ is square-free and that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| < |\mathcal{G}(\mathcal{C})|$. Then, $\mathcal{C}$ contains a modular pointed subcategory of dimension $[\mathcal{G}(\mathcal{C}) : \mathcal{G}(\mathcal{C}_{\text{ad}})]$.*

*Proof.* Let the distinct prime factors of $|\mathcal{G}(\mathcal{C})|$ be $p_1, p_2, \ldots, p_k$. If $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 1$, then from the proof of [CP, Lemma 5.2], $\mathcal{C}_{\text{pt}}$ is a modular subcategory of $\mathcal{C}$. Then, since $\text{FPdim}(\mathcal{C}_{\text{pt}}) = [\mathcal{G}(\mathcal{C}) : \mathcal{G}(\mathcal{C}_{\text{ad}})]$, the claim holds in this case. So, suppose that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| > 1$, and let the prime factors of $|\mathcal{G}(\mathcal{C}_{\text{ad}})|$ be $p_1, p_2, \ldots, p_l$, with $l < k$ (reorder if necessary). From Cauchy's theorem, there must be an invertible $g_i$ of order $p_i$, for $i = 1, \ldots, k$. Consider $g := g_1 \ldots g_l \in \mathcal{G}(\mathcal{C})$ and the fusion subcategory

$$\mathcal{D} \cong \mathcal{C}_{\text{ad}} \oplus \mathcal{C}_g \oplus \mathcal{C}_{g^2} \oplus \cdots \oplus \mathcal{C}_{g^{p_1 \cdots p_l - 1}}.$$

Notice that since $\mathcal{C}_{\text{ad}} \subset \mathcal{D}$, we have that $\mathcal{D}' \subset (\mathcal{C}_{\text{ad}})' \cong \mathcal{C}_{\text{pt}}$ [GN, Corollary 6.9]. From [M, Theorem 3.2], it follows that $\text{FPdim}(\mathcal{D}) = \frac{\text{FPdim}(\mathcal{C})}{p_{l+1}p_{l+2}\cdots p_k}$ and then $\text{FPdim}(\mathcal{D}') = p_{l+1}p_{l+2}\ldots p_k$. So, from Lagrange's theorem, all simple objects in $\mathcal{D}'$, all of which are invertible, have order dividing $p_{l+1}p_{l+2}\ldots p_k$. Then, by Frobenius-Perron dimensions, since $p_1, p_2, \ldots, p_k$ are all relatively prime, $\mathcal{D}' \cap \mathcal{C}_{\text{ad}} = \text{Vec}$. Let $t$ be a simple (then invertible) object in $\mathcal{D} \cap \mathcal{D}'$. Then $t$ is the unit or has a non-trivial order dividing $p_{l+1}p_{l+2}\ldots p_k$. It follows from [N1, Proposition 4.1(i)] that $t$ must be in $\mathcal{C}_{\text{ad}}$, but this implies that $t$ is the unit. Thus, $\mathcal{D} \cap \mathcal{D}' = \text{Vec}$, and so $\mathcal{D}'$ is a modular pointed category of dimension $p_{l+1}p_{l+2}\ldots p_k = [\mathcal{G}(\mathcal{C}) : \mathcal{G}(\mathcal{C}_{\text{ad}})]$, as desired. □

**Proposition 4.17.** *Let $\mathcal{C}$ be an odd-dimensional MTC and suppose that $|\mathcal{G}(\mathcal{C})| = p^2$ for an odd prime $p$ and that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| < |\mathcal{G}(\mathcal{C})|$. If $\mathcal{G}(\mathcal{C}) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $\mathcal{C}$ contains a non-trivial pointed modular subcategory of dimension $[\mathcal{G}(\mathcal{C}) : \mathcal{G}(\mathcal{C}_{\text{ad}})]$.*

*Proof.* If $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 1$, then from the proof of [CP, Lemma 5.2], $\mathcal{C} \cong \mathcal{C}_{\text{ad}} \boxtimes \mathcal{C}_{\text{pt}}$ where $\mathcal{C}_{\text{pt}}$ is a modular subcategory of $\mathcal{C}$. Since $\text{FPdim}(\mathcal{C}_{\text{pt}}) = [\mathcal{G}(\mathcal{C}) : \mathcal{G}(\mathcal{C}_{\text{ad}})]$, the claim holds in this case.

So, suppose that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| > 1$, and thus, we have that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = p$. Then, from [CGP, Proposition 3.17], we obtain that exactly $p$ components contain invertible objects. More precisely, each of them contains $p$ invertibles. Since $\mathcal{G}(\mathcal{C}) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, we have that there are $p + 1$ distinct subgroups of order $p$. Then, there must be a fusion subcategory

$$\mathcal{D} \cong \mathcal{C}_{\text{ad}} \oplus \mathcal{C}_g \oplus \mathcal{C}_{g^2} \oplus \cdots \oplus \mathcal{C}_{g^{p-1}},$$

with $\mathcal{G}(\mathcal{D}) \cong \mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ and $g$ of order $p$. Since $\mathcal{C}_{\mathrm{ad}} \subset \mathcal{D}$, we have that $\mathcal{D}' \subset (\mathcal{C}_{\mathrm{ad}})' \cong \mathcal{C}_{\mathrm{pt}}$ [GN, Corollary 6.9]. From [M, Theorem 3.2], it follows that $\mathrm{FPdim}(\mathcal{D}) = \frac{\mathrm{FPdim}(\mathcal{C})}{p}$ and then $\mathrm{FPdim}(\mathcal{D}') = p$. If $\mathcal{D} \cap \mathcal{D}' = \mathrm{Vec}$, then the claim holds.

So, suppose that $\mathcal{D} \cap \mathcal{D}'$ is not trivial, and we have $\mathcal{D} \cap \mathcal{D}' = \mathcal{D}'$. So, since by construction, $\mathcal{C}_{g^i}$ does not contain any invertibles for $i = 1, 2, \ldots, p-1$, we must have that $\mathcal{D}' \subseteq \mathcal{C}_{\mathrm{ad}}$. Then, from [EGNO, Theorem 8.21.1(i)], we have that

$$\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}} \cap \mathcal{D}')\,\mathrm{FPdim}(\mathcal{D}) = \mathrm{FPdim}(\mathcal{D} \cap (\mathcal{C}_{\mathrm{ad}})')\,\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}).$$

Since $p = \mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}} \cap \mathcal{D}') = \mathrm{FPdim}(\mathcal{G}(\mathcal{C}_{\mathrm{ad}})) = \mathrm{FPdim}(\mathcal{G}(\mathcal{D})) = \mathrm{FPdim}(\mathcal{D} \cap \mathcal{C}_{\mathrm{pt}}) = \mathrm{FPdim}(\mathcal{D} \cap (\mathcal{C}_{\mathrm{ad}})')$ [GN, Corollary 6.9] and $\mathrm{FPdim}(\mathcal{D}) = \frac{\mathrm{FPdim}(\mathcal{C})}{p} = p\,\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$, we get a contradiction and the proposition is proven. ☐

## 5. Solvability

Feit-Thompson's Theorem states that groups of odd order are solvable, and it can be generalized to the case of groups of order not divisible by 4. We will use this result in the following lemma that generalizes [NP, Proposition 7.1], which is a Feit-Thompson Theorem for odd-dimensional weakly group-theoretical fusion categories. Our proof is similar to that of [NP, Proposition 7.1] but we include it for completeness.

**Lemma 5.1.** *Let $\mathcal{C}$ be a weakly group-theoretical fusion category with dimension congruent to 2 modulo 4. Then, $\mathcal{C}$ is solvable.*

*Proof.* Since $\mathcal{C}$ is a weakly group-theoretical fusion category, $\mathcal{C}$ is Morita equivalent to a nilpotent fusion category. By [ENO2, Proposition 4.5], it is enough to show the claim for when $\mathcal{C}$ is a nilpotent fusion category. Let us now prove the proposition by induction on $\mathrm{FPdim}(\mathcal{C})$.

If $\mathrm{FPdim}(\mathcal{C}) = 2$, $\mathcal{C}$ is pointed and then solvable. So, suppose that $\mathrm{FPdim}(\mathcal{C}) > 2$. Since $\mathcal{C}$ is weakly group-theoretical, it is a $G$-extension of a weakly group-theoretical fusion subcategory $\mathcal{C}_1$ for a non-trivial group $G$. Therefore,

$$\mathrm{FPdim}(\mathcal{C}) = |G|\,\mathrm{FPdim}(\mathcal{C}_1) \text{ with } |G| > 1.$$

If $|G|$ is odd, then $\mathrm{FPdim}(\mathcal{C}_1) < \mathrm{FPdim}(\mathcal{C})$ is also congruent to 2 modulo 4 and must be solvable by the inductive hypothesis. Then $\mathcal{C}$ is solvable because it is a $G$-extension of $\mathcal{C}_1$, see [ENO2, Proposition 4.5].

If $|G|$ is even, it is congruent to 2 modulo 4. Then $\mathrm{FPdim}(\mathcal{C}_1)$ is odd and $\mathcal{C}_1$ must be solvable by [NP, Proposition 7.1]. Since $|G|$ is congruent to 2 modulo 4, from the generalized Feit-Thompson's Theorem, $G$ is solvable as well. So, it follows from [ENO2, Proposition 4.5] that $\mathcal{C}$ is solvable. ☐

In particular, a direct consequence of this result is that if $\mathcal{C}$ is an MTC with dimension not divisible by 4 that is an extension of a pointed category then it must be solvable.

**Corollary 5.2.** *Let $\mathcal{C}$ be an MTC of dimension not divisible by 4. If $\mathcal{C}_{\mathrm{ad}} \subseteq \mathcal{C}_{\mathrm{pt}}$, then $\mathcal{C}$ is solvable. So, if $\mathcal{C}$ is not solvable, all components containing invertible objects must contain at least one non-invertible simple object.*

*Proof.* Since $\mathcal{C}$ is a $\mathcal{G}(\mathcal{C})$-extension of a pointed category, namely $\mathcal{C}_{\mathrm{ad}}$, we have that $\mathcal{C}$ is nilpotent. So, by definition, $\mathcal{C}$ is weakly group-theoretical and, as a consequence of Lemma 5.1 and [NP, Proposition 7.1], $\mathcal{C}$ is solvable. The second comment follows directly from [CGP, Proposition 3.17]. □

Recall that it was conjectured in [CP, Conjecture 1.2] that any odd-dimensional modular category is solvable. We conjecture this is true more generally for modular categories with dimension not divisible by 4 because of the following corollary.

The corollary below follows immediately from Theorem 3.9 and [ENO2, Proposition 4.5].

**Corollary 5.3.** *Odd-dimensional MTCs are solvable if and only if MTCs of dimension congruent to 2 modulo 4 are solvable.*

This then leads us to generalize Conjecture 1.2 in [CP] as the following.

**Conjecture 5.4.** *MTCs of Frobenious-Perron dimension not divisible by 4 are solvable.*

## 6. Algorithms

Let $\mathcal{C}$ be an MTC of dimension not divisible by 4. In this section, we will provide 2 different algorithms to compute all possible arrays of Frobenius-Perron dimensions of simple objects in a component of the universal grading. This could help classify odd-dimesional MTCs of a fixed rank, which would lead to the classification of MTCs of dimension congruent to 2 modulo 4 as well, see Section 3.

For the rest of this section, given a component $\mathcal{C}_g$, let $I_g$ and $N_g$ denote the number of invertible and non-invertible simple objects in $\mathcal{C}_g$, respectively. Also, for a simple object $X$ let $m_X$ be as in Equation (5). The following algorithms are similar to the ones in [BR], in the sense that they both study types of Egyptian fraction decompositions.

6.1. **Algorithm 1.** Our first algorithm will compute all possible arrays of dimensions of simple objects in $\mathcal{C}_g$, given that we know the numbers $N_g, I_g, |\mathcal{G}(\mathcal{C})|$ and $m_X$ for some simple object $X \in \mathcal{C}$. Although we rarely know the value of $m_X$ for a simple object $X$ which could make this impractical, this algorithm will be especially useful when we know the ranks of all the components of the universal grading and one turns out to be significantly smaller than the rest (e.g. if there is a $g$ for which $\mathrm{rank}(\mathcal{C}_g) = 1$, then we know that $m_X = |\mathcal{G}(\mathcal{C})|$ for the simple object $X \in \mathcal{C}_g$).

We will now describe Algorithm 1. We will need the following definition.

**Definition 6.1.** A *modified Egyptian Fraction decomposition of order* $p$ of a fraction $\frac{m}{n}$ is a set of fractions $\{\frac{1}{n_1^2}, \frac{1}{n_2^2}, \dots, \frac{I}{r^2 n_p^2}\}$ such that $I, r \in \mathbb{Z}_{>0}$, $n_i$ is odd for all $1 \le i \le p$, and

$$\frac{m}{n} = \frac{1}{n_1^2} + \frac{1}{n_2^2} + \ldots + \frac{1}{n_{p-1}^2} + \frac{I}{r^2 n_p^2}.$$

We denote such a decomposition as $[n_1, n_2, \dots, n_{p-1}, r, I]$, and assume that $n_1 \le \ldots \le n_{p-1}$.

**Remark 6.2.** In the above definition, $\gcd(I, r^2 n_p^2)$ is not necessarily 1.

Let $\mathcal{C}$ be an MTC and $g \in \mathcal{G}(\mathcal{C})$. For a simple object $X \in \mathcal{C}$ let $m_X$ be as in Equation (5).

**Lemma 6.3.** *Suppose that all simple objects in $\mathcal{C}$ have odd Frobenius-Perron dimension, and that for some $X \in \mathcal{O}(\mathcal{C})$, $m_X = s^2 a$ for $a, s \in \mathbb{Z}_{>0}$ such that $a$ is square-free. Then, to find all possible arrays of Frobenius-Perron dimensions of simple objects in $\mathcal{C}_g$ it is sufficient to find all modified Egyptian fraction decompositions of $\frac{a}{|\mathcal{G}(\mathcal{C})|}$ of order $N_g + 1$ with $r = s$ and $I = I_g$.*

*Proof.* Let $d_1, d_2, \dots, d_{N_g}$ denote the Frobenius-Perron dimensions of the non-invertible simple objects in $\mathcal{C}_g$, and $d_X$ the Frobenius-Perron dimension of $X$. Then

$$\mathrm{FPdim}(\mathcal{C}) = m_X d_X^2 = s^2 a d_X^2 = |\mathcal{G}(\mathcal{C})|(I_g + d_1^2 + d_2^2 + \ldots + d_{N_g}^2),$$

or equivalentely,

$$\frac{a}{|\mathcal{G}(\mathcal{C})|} = \frac{I_g}{s^2 d_X^2} + \frac{d_1^2}{s^2 d_X^2} + \ldots + \frac{d_{N_g}^2}{s^2 d_X^2}.$$

Since $d_1^2, d_2^2, \dots, d_{N_g}^2$ all divide $\mathrm{FPdim}(\mathcal{C}) = m_X d_X^2 = s^2 d_X^2 a$ and $a$ is square-free, we have that they all divide $s^2 d_X^2$. Notice that since $\mathrm{FPdim}(\mathcal{C}) \equiv 2 \pmod 4$, we must have that $s$ is odd. Hence, there exist odd integers $n_1, n_2, \dots, n_{N_g}$ such that $s^2 d_X^2 = n_i^2 d_i^2$ for all $1 \le i \le N_g$. So, we get that

$$\frac{a}{|\mathcal{G}(\mathcal{C})|} = \frac{I_g}{s^2 d_X^2} + \frac{1}{n_1^2} + \frac{1}{n_2^2} + \ldots + \frac{1}{n_{N_g}^2}, \tag{7}$$

as desired. □

**Lemma 6.4.** *Let $\mathcal{C}$ be an integral MTC. Let $X \in \mathcal{O}(\mathcal{C})$ and suppose that $m_X = s^2 a$ for $a, s \in \mathbb{Z}_{>0}$ such that $a$ is square-free. If we know that* $\mathrm{FPdim}(X)$ *is bounded below by some $b \in \mathbb{Z}_{>0}$, then for any modified Egyptian Fraction decomposition $[n_1, n_2, \dots, n_{N_g}, s, I_g]$ of $\frac{a}{|\mathcal{G}(\mathcal{C})|}$ we have that*

$$\sqrt{\frac{|\mathcal{G}(\mathcal{C})|}{a}} < n_1 \le bs\sqrt{\frac{N_g |\mathcal{G}(\mathcal{C})|}{b^2 a s^2 - |\mathcal{G}(\mathcal{C})| I_g}}.$$

*Proof.* We use the same notation as in the proof of Lemma 6.3. Notice that

$$\frac{a}{|\mathcal{G}(\mathcal{C})|} - \frac{1}{n_1^2} > 0,$$

and so, $n_1 > \sqrt{\frac{|\mathcal{G}(\mathcal{C})|}{a}}$, and the lower bound holds. On the other hand, since $n_1 \leq \ldots \leq n_{N_g}$, from Equation 7 we can see that

$$\frac{a}{|\mathcal{G}(\mathcal{C})|} \leq \frac{I_g}{s^2 d_X^2} + \frac{N_g}{n_1^2}.$$

Thus, since $d_X \geq b$, we have that

$$\frac{b^2 a s^2 - I_g |\mathcal{G}(\mathcal{C})|}{|\mathcal{G}(\mathcal{C})| s^2 b^2} \leq \frac{N_g}{n_1^2},$$

and the upper bound follows after re-arranging this inequality. □

**Algorithm 1.** *The first algorithm consists of the following steps.*

- *Initialize* $m = 3$, $k = N_g$, $gc = |\mathcal{G}(\mathcal{C})|, lc = 1$, *and* $l = m_X$.
- *Find the largest square divisor* $s$ *of* $l$. *Set* $d1 = \frac{l}{s}$ *and* $d2 = gc$, *and make* $d1$ *and* $d2$ *relatively prime if necessary.*
- *Compute the minimum and maximum values of* $n_1$ *as per Lemma 6.4 with* $b = m$.
- *For each possible* $n_1$ *in said range, update* $lc$ *to be the lcm of the current lcm and* $n_1$, *update* $d1$ *and* $d2$ *to be* $d1n_1^2 - d2$ *and* $d_2 n_1^2$ *respectively (make* $d1$ *and* $d2$ *relatively prime if necessary). Also, update* $m$ *to be the floor of* $\frac{lc}{s}$. *If the denominator of the upper bound in Lemma 6.4 is negative with this value of* $m$, *then multiply* $m$ *by the appropriate odd integer to make it positive. Decrease* $k$ *by 1.*
- *Until* $k = 0$, *keep performing the last two steps by calling the recursive method again. Also, when computing the minimum value of* $n_i$, *take the maximum of the one given in Lemma 6.4 and* $n_{i-1}$ *(since* $n_1 \leq n_2 \ldots \leq n_{N_g}$*).*
- *When* $k = 0$, *test whether* $f = \frac{I_g d2}{s d1}$ *is a perfect square. If it is, then check whether* $fs$ *is divisible by* $lc^2$. *Lastly, check whether any of* $n_1, n_2, \ldots, n_{N_g}$ *are equal to* $\sqrt{sf}$ *to avoid more than* $I_g$ *invertible objects.*
- *Return the solution.*

This algorithm computes all modified Egyptian Fraction decompositions of the form $[n_1, n_2, \ldots, n_{N_g}, s, I_g]$ of $\frac{a}{|\mathcal{G}(\mathcal{C})|}$, and thus by Lemma 6.3 from this we can get all possible arrays of Frobenius-Perron dimensions of the simple objects in $\mathcal{C}_g$. The code we used to implement Algorithm 1 can be found in Appendix A.

6.2. **Algorithm 2.** Our second algorithm will also compute the possible dimensions of simple objects in a component $\mathcal{C}_g$, but in this case we will need to know the numbers $\operatorname{rank}(\mathcal{C}), N_g, I_g$, and $|\mathcal{G}(\mathcal{C})|$ instead.

The following proof is similar to [CGP, Lemma 3.1].

**Lemma 6.5.** *Let* $\mathcal{C}$ *be an MTC for which all simple objects have odd Frobenius-Perron dimension, and let* $X$ *be the simple object of greatest Frobenius-Perron dimension in* $\mathcal{C}_g$. *Then,*

- $|\mathcal{G}(\mathcal{C})| < m_X < N_g + \frac{I_g}{9} + 1$.
- $m_X \equiv \operatorname{rank}(\mathcal{C}) \pmod 8$,

*where* $m_X$ *is as in Equation* (5)*.*

*Proof.* Let $d_1$ denote the Frobenius-Perron dimension of $X$, and let $d_2 \geq d_3, \ldots \geq d_{N_g}$ be the Frobenius-Perron dimensions of the remaining non-invertible simple objects in $\mathcal{C}_g$. Since $d_1 \geq 3$, we have

$$m_X d_1^2 = |\mathcal{G}(\mathcal{C})|(I_g + d_1^2 + \ldots + d_{N_g}^2) \leq |\mathcal{G}(\mathcal{C})|(I_g + N_g d_1^2) < |\mathcal{G}(\mathcal{C})| \left(N_g + \frac{I_g}{9} + 1\right) d_1^2,$$

and so, the upper bound follows. The lower bound immediately follows from the above equation as well. Notice also that

$$m_X \equiv m_X d_1^2 \equiv \operatorname{FPdim}(\mathcal{C}) \equiv \operatorname{rank}(\mathcal{C}) \pmod 8,$$

and so, the congruence follows as well. □

**Algorithm 2.** *Our second algorithm consists of the following steps.*

- *From Lemma 6.5, compute the minimum and maximum value for* $m_X = i$ *where* $X$ *is the simple object of greatest dimension in* $\mathcal{C}_g$*.*
- *For each value of* $i$*, find the largest square divisor* $s$ *of* $i$*, and initialize* $d1 = i - g_c$ *and* $d2 = g_c s$*. Make* $d1$ *and* $d2$ *relatively prime if necessary. Initialize* $k$ *to be* $N_g - 1$*.*
- *Use Algorithm 1 to find the possible solutions for this particular value of* $i$*.*
- *Return the solution.*

This algorithm is a generalization of the previous one: we are finding the minimum and maximum value of $m_X$, fixing the value in each case, and then using Algorithm 1.

## 7. Future Work

We are working on a proof of Proposition 3.7 using irreducible projective representations, as it will provide a completely different viewpoint on why the claim holds.

## Appendix A. Code

We present here our code for Algorithms 1 and 2. They were coded in C++. Because of the maximum value of a long long int in C++ and the exponential growth of the number of solutions as the rank increases, this code is unable to compute dimensions of simple objects in components of large rank.

Listing 1. Code for Algorithm 1

```
#include <algorithm>
#include <iostream>
#include <algorithm>
#include <random>
#include <cmath>
using namespace std;
static int c = 0;
static int b = 0;
vector<vector<long long int> > v;
     long long int square_d( long long int n)
    {
     long long int d = 1;
    for (int i = 3; i <= sqrt(n); i = i+2)
    {
        while ((n % (i*i)) == 0)
        {
            d = d * i * i;
            n = n/(i*i);
        }
    }
    return d;
    }
    bool isPrime(long long int n)
    {
        if (n <= 1)
            return false;
        if (n % 2 == 0)
            return false;
        for (int i = 3; i * i <= n; i += 2)
            if (n % i == 0)
                return false;
        return true;
    }
    bool isSquare(long long int n)
    {
        if (square_d(n) == n)
            return true;
        return false;
```

```
}
 long long int gcd(long long int a, long long int b)
{
  if (b == 0)
    return a;
  return gcd(b, fmod(a,b));
}
long long int lcm( long long int a,  long long int b)
{
    return a*b/gcd(a,b);
}
     void possible_solution( vector<long long int> sol, long long int lc, long long int s, long long int d1, long long int d2, long long int k, long long int g, long long int m)
{
    if (k == 0)
    {
        int d = 0;
        if (((g*d2) % (s*d1)) == 0)
        {
            long long int f = g*d2/(s*d1);
            if (isSquare(f) && (((s*f) % (lc*lc)) == 0))
            {
               sol.push_back(sqrt(f));
               for (int i = 0; i < sol.size(); i++)
                   if (sol.at(i) == sqrt(s*f))
                       d++;
          if ((s == 1 && d == 1) || (s != 1 && d == 0))
            {
               int e = 0;
               for (int i = sol.size() - 2; i >= 0; i--)
               {
                    sol[i] = sol.at(sol.size() - 1)*sqrt(s)/(sol.at(i);
               }
               sort(sol.begin(), sol.end());
               for (int i = 0; i < c; i++)
               {
                   if (sol == v[i])
                       e++;
               }
                if (e == 0)
                {
                    c++;
                   for (int i = 0; i < sol.size(); i++)
               {
                    cout << sol.at(i) << " ";
               }
```

```
                        cout << endl;
                        v.push_back(sol);
                    }
                }
                }
            }
        }
        else
        {
         long long int mi;
          if (sol.size() == 0)
              mi = sqrt(d2/d1) + 1;
            else
              mi = std::max((long long int)(sqrt(d2/d1) + 1), (sol.at(sol.size() - 1)
                  ));
              double m1 = ((double) lc/(double) sqrt(s));
              m = floor(m1);
                if (m % 2 == 0)
                    m += 1;
             double d = m*m*d1*s - d2*g;
            if (d <= 0)
            {
             long long int st = sqrt(d2*g/(d1*s)) + 1;
              long long int l = floor(st/m) + 1;
                   if (l % 2 == 0)
                       l += 1;
               m *= (long long int) l;
             d = m*m*d1*s - d2*g;
            }
          double maxx = sqrt((double) m*m*s*k*d2/(double) d) + 2;
       if (mi % 2 == 0)
            mi += 1;
        for (long long int i = mi; i <= maxx; i +=2)
        {
            sol.push_back(i);
            long long int w = lc;
            lc = lcm(lc, i);
             long long int k1 = d1*i*i - d2;
             long long int k2 = d2*i*i;
             long long int gc = gcd(k1, k2);
             long long int l1 = k1/gc;
             long long int l2 = k2/gc;
            possible_solution(sol, lc, s, l1, l2, k-1, g, m);
            sol.pop_back();
            lc = w;
        }
    }
```

```
        }
int main()
{
 vector< long long int> sol;
 long long int lc = 1;
  long long int m = 3;
 long long int l, g_c, k, g;
 long long int gc = gcd(g_c, l/square_d(l));
 long long int d1 =  l/(square_d(l)*gc);
 long long int d2 = g_c/gc;
    possible_solution(sol, lc, square_d(l), d1, d2, k, g, m);
}
```

Listing 2. Code for Algorithm 2

```
#include <iostream>
#include <algorithm>
#include <random>
#include <gmp.h>
#include <cmath>
using namespace std;
static int c = 0;
static int b = 0;
vector<vector<long long int> > v;
     long long int square_d( long long int n)
    {
     long long int d = 1;
    for (int i = 3; i <= sqrt(n); i = i+2)
    {
        while ((n % (i*i)) == 0)
        {
            d = d * i * i;
            n = n/(i*i);
        }
    }
    return d;
    }
    bool isSquare( long long int n)
    {
        long long int s = sqrt(n);
       return (s * s == n);
    }
     long long int gcd( long long int a,  long long int b)
    {
      if (b == 0)
        return a;
```

```
    return gcd(b, a%b);
  }
   long long int lcm( long long int a,  long long int b)
  {
      return a*b/gcd(a,b);
  }
  void possible_solution( vector<long long int> sol, long long int lc, long long int s, long long int d1, long long int d2, long long int k, long long int g, long long int m)
  {
      if (k == 0)
      {
          int d = 0;
          if (((g*d2) % (s*d1)) == 0)
          {
              long long int f = g*d2/(s*d1);
              if (isSquare(f) && (((s*f) % (lc*lc)) == 0))
              {
                 sol.push_back(sqrt(f));
                 for (int i = 0; i < sol.size(); i++)
                     if (sol.at(i) == sqrt(s*f))
                         d++;
            if ((s == 1 && d == 1) || (s != 1 && d == 0))
              {
                 int e = 0;
                 for (int i = sol.size() - 2; i >= 0; i--)
                 {
                      sol[i] = sol.at(sol.size() - 1)*sqrt(s)/(sol.at(i);
                 }
                 sort(sol.begin(), sol.end());
                 for (int i = 0; i < c; i++)
                 {
                     if (sol == v[i])
                         e++;
                 }
                  if (e == 0)
                  {
                      c++;
                     for (int i = 0; i < sol.size(); i++)
                 {
                      cout << sol.at(i) << " ";
                 }
                      cout << endl;
                      v.push_back(sol);
                  }
              }
              }
```

```
            }
        }
        else
        {
         long long int mi;
          if (sol.size() == 0)
              mi = sqrt(d2/d1) + 1;
            else
              mi = std::max((long long int)(sqrt(d2/d1) + 1), (sol.at(sol.size() - 1)));
              double m1 = ((double) lc/(double) sqrt(s));
              m = floor(m1);
                if (m % 2 == 0)
                    m += 1;
             double d = m*m*d1*s - d2*g;
            if (d <= 0)
            {
             long long int st = sqrt(d2*g/(d1*s)) + 1;
              long long int l = floor(st/m) + 1;
                  if (l % 2 == 0)
                      l += 1;
               m *= (long long int) l;
             d = m*m*d1*s - d2*g;
            }
          double maxx = sqrt((double) m*m*s*k*d2/(double) d) + 2;
       if (mi % 2 == 0)
            mi += 1;
        for (long long int i = mi; i <= maxx; i +=2)
        {
            sol.push_back(i);
            long long int w = lc;
            lc = lcm(lc, i);
             long long int k1 = d1*i*i - d2;
             long long int k2 = d2*i*i;
             long long int gc = gcd(k1, k2);
             long long int l1 = k1/gc;
             long long int l2 = k2/gc;
            possible_solution(sol, lc, s, l1, l2, k-1, g, m);
            sol.pop_back();
            lc = w;
        }
    }
        }
int main()
{
 vector< long long int> sol;
 long long int lc = 1;
```

```
long long int m = 3;
long long int k, g, g_c, rank;
long long int mod = fmod(rank, 8);
double ma = g_c*(k + g/9.0) + 1;
for (int i = mod; i <= ma; i += 8)
{
    if (a > 0)
    {
       long long int d1 = i - g_c;
       long long int d2 = g_c*square_d(i);
       long long int gc = gcd(d1, d2);
       d1 = d1/gc;
       d2 = d2/gc;
       possible_solution(sol, lc, square_d(i), d1, d2, k-1, g, m);
    }
}
}
```

## References


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