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Research Report

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Title of Research Report

Standard modules of the Temperley-Lieb algebra at zero

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STANDARD MODULES OF THE TEMPERLEY-LIEB ALGEBRA AT ZERO

EDDY LI

Abstract. For any $\beta, q \in \mathbb{C}$ such that $\beta = q^{1/2} + q^{-1/2}$, the Hecke algebra $\mathcal{H}_n(q)$ surjects onto the Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$. When q is not a root of unity, the irreducible modules of $\mathsf{TL}_n(\beta)$ are exactly the standard modules W_ℓ^n. However, the standard modules cease to be irreducible when q is a root of unity, most notably when $\beta = 0$. In this paper, we show that the category of representations of $\mathsf{TL}_n(0)$ for even n is equivalent to the category of the representations of the quotient of a straight-line quiver by some ideal. This implies the existence of a long exact sequence on the standard modules. Moreover, we construct explicit homomorphisms between adjacent modules and demonstrate exactness everywhere. The surjection from the Hecke algebra also implies the existence of a long exact sequence among the Specht modules on partitions having at most two rows, whose structure resolves nicely over characteristic two. We end by observing a connection between our results and the task of computing the Jones polynomial of a braid closure at $t = -1$.

Keywords. Temperley-Lieb algebra, Hecke algebra, standard modules, projective modules, quiver algebra, perverse sheaves, long exact sequence, composition series, Specht modules, Specht polynomials, Young symmetrizer, Ocneanu trace, Jones polynomial.

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Name of supervising teacher: Kenta Suzuki

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1. Introduction

The Temperley-Lieb algebra was formulated by the mathematical physicists Temperley and Lieb in 1971 in their study of planar lattice models to generalize the transfer-matrix method for problems of percolation and coloring [32]. Since its conception, the Temperley-Lieb algebra was quickly recognized for its significance as an aid to the study of ice-type and Potts models [2]. It has also played a major role in mathematical physics due to its numerous applications to integrable models, quantum groups, and von Neumann algebras [1]. About a decade later, the eminent knot theorist Jones independently rediscovered the Temperley-Lieb relations in his work on finite-index von Neumann subfactors, which inspired the creation of the Jones polynomial [18]. The curious reader who wishes to know more about the origins and early developments of the Temperley-Lieb algebra may desire to consult the survey [10].

Following the work of Jones [18, 19, 20], the Temperley-Lieb algebra has generated increasing interest from the perspective of representation theory. For instance, analyzing the structure of the irreducible and indecomposable representations of the Temperley-Lieb algebra has proven to be a rather nontrivial task [16, 28, 33]. Additionally, the Temperley-Lieb algebra is connected to several other algebraic objects, including the Hecke algebra and the braid group [20, 33], as well as certain closely related variants, such as its affine or nilpotent deformations [3]. Notably, it follows that the Temperley-Lieb algebra is related to the double-affine Hecke algebra, also known as the Cherednik algebra, albeit perhaps more indirectly [4].

1.1. **Main results.** The Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$ depends on an index $n \in \mathbb{N}$ and a complex parameter $\beta \in \mathbb{C}$. There exist certain standard modules W_ℓ^n indexed by n and ℓ, where ℓ is a nonnegative integer of the same parity as n. Let $q \in \mathbb{C} \setminus \{0\}$ satisfy $\beta = q^{1/2} + q^{-1/2}$. It is not difficult to construct a surjection from the Hecke algebra $\mathcal{H}_n(q)$ onto the Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$, implying that the representation of the Specht module $S^{((n+\ell)/2,(n-\ell)/2)}$ factors through to that of the standard module W_ℓ^n. For generic β, in which q is not a root of unity, Westbury proved by computing the Gram determinants on each standard module that the irreducible modules of $\mathsf{TL}_n(\beta)$ are exactly the standard modules and moreover that $\mathsf{TL}_n(\beta)$ is semisimple [34].

When q is a root of unity, as is the case for specializations such as $\beta \in \{0, 1, \sqrt{2}, \frac{1+\sqrt{5}}{2}, \sqrt{3}, 2\}$, the structure of the Temperley-Lieb algebra becomes much less straightforward. For such values of β, the Temperley-Lieb algebra ceases to be semisimple. Goodman and Wenzl applied the algebraic methods of evaluation at critical parameter values and spectral analysis for idempotents to obtain the block decomposition and dimensionality of the irreducible modules of $\mathsf{TL}_n(\beta)$ [16]. The investigations of Goodman and Wenzl have generated further interest in the structure of the Temperley-Lieb algebra at $\beta = q^{1/2} + q^{-1/2}$ for q a root of unity [12, 22, 23, 25, 28, 29]. For the noteworthy case of $\beta = 0$, of particular importance is the work of Ridout and Saint-Aubin, in which categorical methods are used to demonstrate that, for odd n, the standard modules remain irreducible, while for even n they exhibit composition series of length two, entailing a unique irreducible quotient and irreducible submodule [28]. In this paper, we shall focus extensively on the specialization to $\beta = 0$, which is important as it corresponds to the value of $q = -1$ of a low multiplicative order of 2.

Our first main result demonstrates an equivalence between the category of representations of $\mathsf{TL}_n(0)$ and the category of representations of the path algebra $\mathbb{C}\mathcal{Q}_{n/2}$ on the straight-line quiver of $\frac{n}{2}$ vertices quotiented by some ideal J. Our investigations are motivated upon the work of Ridout and Saint-Aubin, in which they define a sequence of projective principal indecomposable modules P_ℓ^n on $\mathsf{TL}_n(0)$ [28]. For even n, we show that $\mathsf{TL}_{n-1}(0)$ is semisimple and compute the dimension of the homomorphism spaces among the P_ℓ^n and W_ℓ^n. We observe that the dimensionality of our computed homomorphism spaces closely resembles the number of distinct paths on the double-sided straight-line quiver $\mathcal{Q}_{n/2}$ on $\frac{n}{2}$ vertices satisfying certain special conditions. Then, treating the projective and standard modules as intrinsically being objects, it is natural to explore the relationship between $\mathsf{TL}_n(0)$ and $\mathbb{C}\mathcal{Q}_{n/2}$ from a categorical perspective, leading us to construct the ideal $J \subseteq \mathbb{C}\mathcal{Q}_{n/2}$. Now we state our first main result as shown below.

Theorem 1.1. *There exists an ideal J of the path algebra $\mathbb{C}\mathcal{Q}_{n/2}$ for which the functor from $\mathbf{\Phi}\colon \mathbf{Rep}(\mathsf{TL}_n(0)) \to \mathbf{Rep}(\mathbb{C}\mathcal{Q}_{n/2}/J)$ given by*

$$\mathbf{\Phi}(X) = \ \mathrm{Hom}(P_2^n, X) \rightleftarrows \mathrm{Hom}(P_4^n, X) \rightleftarrows \mathrm{Hom}(P_6^n, X) \rightleftarrows \cdots \rightleftarrows \mathrm{Hom}(P_n^n, X)$$

establishes a category equivalence $\mathbf{Rep}(\mathsf{TL}_n(0)) \simeq \mathbf{Rep}(\mathbb{C}\mathcal{Q}_{n/2}/J)$.

Motivated by the implications of Theorem 1.1, our second main result establishes an explicit sequence of homomorphisms that give rise to a long exact sequence on the standard modules of $\mathsf{TL}_n(0)$. Indeed, given our homomorphism space computations on the projective and standard modules, we may evaluate the functor $\mathbf{\Phi}$ upon each standard module to obtain the objects $\mathbf{\Phi}(W_\ell^n)$, which by Theorem 1.1 can be verified to inherit the structure of a long exact sequence over the quotient of $\mathbb{C}\mathcal{Q}_{n/2}/J$. Projectivity implies the exactness of $\mathbf{\Phi}$, and hence it is automatic that a long exact sequence on the standard modules must necessarily exist. More prominently, we can do even better by deriving explicit diagrammatic formulations of the homomorphisms connecting adjacent standard modules in the long exact sequence. The upshot is that we have consequentially established complete diagrammatic descriptions of all irreducible modules of $\mathsf{TL}_n(0)$ for all n. In particular, the following theorem encapsulates our second main result.

Theorem 1.2. *Let n be even. There exist maps $\phi_\ell^n\colon W_{\ell+2}^n \to W_\ell^n$ for all nonnegative even ℓ such that the sequence*

$$0 \xrightarrow{\phi_n^n} W_n^n \xrightarrow{\phi_{n-2}^n} W_{n-2}^n \xrightarrow{\phi_{n-4}^n} \cdots \xrightarrow{\phi_2^n} W_2^n \xrightarrow{\phi_0^n} W_0^n \longrightarrow 0$$

is exact. Moreover, the collection $\{\operatorname{im}\phi_\ell^n \mid 0 \le \ell \le n-2, \ell \equiv 0 \pmod 2\}$ constitutes a complete set of distinct irreducible modules of $\mathsf{TL}_n(0)$.

Our third main result lifts the long exact sequence on the standard modules that we have discovered in Theorem 1.2 onto a long exact sequence on the q-Specht modules specialized to $q = -1$ over characteristic two. We motivate the long exact sequence on such a collection of (-1)-Specht modules by observing that it is guaranteed to exist. This is because, as established by James and Mathas, the closest analogues of the standard modules over $\mathcal{H}_n(q)$ are the q-Specht modules [17]. For $\beta = q^{1/2} + q^{-1/2}$, recall that the representation of the q-Specht modules on two-row partitions factors through, via the aforementioned surjection from $\mathcal{H}_n(q)$ to $\mathsf{TL}_n(\beta)$, onto the representation due to the standard modules of the Temperley-Lieb algebra. Hence, a homomorphism between standard modules must also intertwine with the action of $\mathcal{H}_n(q)$ when interpreted as a linear map between q-Specht modules. As a result, the long exact sequence given by Theorem 1.2 under specialization to $\beta = 0$ automatically implies the existence of a long exact sequence of (-1)-Specht modules, providing clear motivation for our foray into the realm of Specht modules as we investigate the structure of such a long exact sequence.

Unfortunately, for $q \neq 1$, the q-Specht modules remain fairly intractable objects in comparison to the Specht modules or standard modules. In contrast, when $q = 1$, the q-Specht modules and Hecke algebra collapse respectively into the Specht module and the symmetric group algebra. This endows Specht modules with a vastly simpler structure that may even be expressed as spans of polynomials. We can make our analysis for the specialization $q = -1$ much more tractable by working over a base field in which -1 and 1 are in fact equal, in which case we are equivalently analyzing the structure of the Specht modules over characteristic two. As our third main result details below, we prove that when we take base field $\mathbb{k} = \mathbb{F}_2$ and quotient out all square terms, the long exact sequence on these (-1)-Specht modules becomes fairly tractable.

Theorem 1.3. *Let n be even, and let T^λ be the polynomial (-1)-Specht modules over the ring $\mathbb{F}_2[x_1, x_2, \ldots, x_n]/\langle x_1^2, x_2^2, \ldots, x_n^2\rangle$. There then exists a long exact sequence*

$$0 \longrightarrow T^{(n)} \longrightarrow T^{(n-1,1)} \longrightarrow \cdots \longrightarrow T^{(n/2+1,n/2-1)} \longrightarrow T^{(n/2,n/2)} \longrightarrow 0$$

where the homomorphism between adjacent modules is multiplication by $\sum_{i=1}^n x_i$.

Lastly, we emphasize that we may motivate the above main results, most notably Theorem 1.3, by computing the Jones polynomial of braid closures. Let $\pi : B_n \to \mathcal{H}_n(q)$ be the natural homomorphism given by mapping generators to generators. Let χ_λ be the character of the q-Specht module indexed by the partition $\lambda \vdash n$ on $\mathcal{H}_n(q)$. Let $t = q$. We can formulate the Jones polynomial as a polynomial linear combination of such characters.

Proposition 1.4. *The Jones polynomial of the closure of any braid $\alpha \in B_n$ is given by*

$$V_{\hat{\alpha}}(t) = \frac{(-1)^{n-1}(\sqrt{t})^{e(\alpha)-n+1}}{1+t} \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{i=k}^{n-k} t^i \right) \chi_{(n-k,k)^\top}(\pi(\alpha)).$$

Observe that at $t = -1$ the denominator $1 + t$ vanishes, implying that the numerator must vanish as well. If n is odd, one can check by counting parities that $\sum_{i=k}^{n-k}(-1)^i = 0$, guaranteeing that the numerator vanishes automatically. However, if instead we have n even, for $t = -1$ observe that $\sum_{i=k}^{n-k}(-1)^i$ does not vanish, instead evaluating to either 1 or -1 based on parity. As a result, by Proposition 1.4 we expect the identity

$$\sum_{k=0}^{n/2} (-1)^k \chi_{(n-k,k)^\top}(\pi(\alpha)) = 0.$$

Tensoring with the sign representation to transform the character due to $S^{(n-k,k)^\top}$ to that of $S^{(n-k,k)}$, we arrive exactly at the vanishing alternating sum which follows as a consequence of the long exact sequence of Specht modules index by two-row partitions as illustrated in Theorem 1.3.

1.2. Future work.

Currently, we are investigating the structure of the standard modules $M(S^\lambda)$ of the Cherednik algebra H_c on type $\mathbf{A}$, where $\lambda \vdash n$ is a partition and c maps every simple reflection of $\mathfrak{S}_n$ to $\frac{1}{2}$. This follows from the fact that, over type $\mathbf{A}$, the map c must be invariant over each simple reflection in which the specialization to c is analogous to specializing the Hecke algebra to $q = e^{2i\pi c}$ [4]. For $q = e^{i\pi} = -1$, the corresponding specialization is to $c = \frac{1}{2}$.

As a result, it would be interesting if there exists yet another long exact sequence analogous to the form of Theorem 1.3, this time on the collection of $M(S^\lambda)$, in which λ again is a two-row partition. Correspondingly, we propose the following question.

Question 1.5. *Does there exist a long exact sequence of standard modules of the form*

$$0 \longrightarrow M(S^{(n)}) \longrightarrow M(S^{(n-1,1)}) \longrightarrow \cdots \longrightarrow M(S^{(n/2+1,n/2-1)}) \longrightarrow M(S^{(n/2,n/2)}) \longrightarrow 0,$$

and if so, what is the structure of the homomorphisms connecting adjacent modules?

Pivoting away from algebra and towards knot theory, the statement of Proposition 1.4 will likewise generate some further research of interest. A link L embedded in the 3-sphere $\mathbb{S}^3$ is said to exhibit a splitting if there exists a subspace $B \subseteq \mathbb{S}^3 \setminus L$ homeomorphic to $\mathbb{S}^2$ such that L intersects both components of $\mathbb{S}^3 \setminus B$ [24]. Colloquially speaking, this occurs exactly when L can be separated into two rigid components that can be moved arbitrarily far away from each other, without disturbing the isotopy class of L. If L is a split link with components L_1 and L_2, then $V_L(t) = (-t^{1/2} - t^{-1/2})V_{L_1}(t)V_{L_2}(t)$, and hence $V_L(-1) = 0$. Thus we wonder if it is possible to classify all non-split n-braids $\alpha \in B_n$ for which $V_{\hat{\alpha}}(-1) = 0$.

For odd n, the singularity at $t = -1$ is removed easily by the polynomial coefficients $\sum_{i=k}^{n-k} t^i$ of the characters, resulting in a clean formula for the Jones polynomial. For instance, when $n = 3$, the author has derived the following result [21].

Theorem 1.6 ([21], Theorem 4.7)**.** *Let $\alpha \in B_3$ satisfy $V_{\hat{\alpha}}(-1) = 0$. Then there exist integers m and k for which $\hat{\alpha} = \hat{\alpha}_0$, where $\alpha_0 = \sigma_2^k(\sigma_1\sigma_2\sigma_1)^{4m}$.*

For even n, the singularity at $t = -1$ cannot be removed in a straightforward fashion. Indeed, although our main results imply that it must be removable, the polynomial coefficients of the characters are not divisible by $1+t$. Hence, the explicit resolution of the singularity at $t = -1$ of $V_{\hat{\alpha}}(t)$ remains unclear. The simplest case is that of $n = 4$. Motivated by Theorem 1.6, it would be interesting to further delve into the behavior of the Specht module characters as t approaches -1. For instance, we can obtain the value of $V_{\hat{\alpha}}(-1)$ by taking the derivative of $(1+t)V_{\hat{\alpha}}(t)$ and specializing to $t = -1$. Hence, we propose the following question.

Question 1.7. *What are the possible 4-braids $\alpha \in B_4$ such that $V_{\hat{\alpha}}(-1) = 0$? More fundamentally, how do we evaluate the derivative of $\chi_\lambda(\pi(\alpha))$ for $\lambda \in \{(1,1,1,1),(2,1,1),(2,2)\}$ at the specialization $t = -1$?*

1.3. **Structure of the paper.** Our paper is organized as follows. In Section 2, we introduce the Temperley-Lieb algebra, the Hecke algebra, and the braid group. In Section 3.1, we introduce the irreducible and projective modules on $\mathsf{TL}_n(0)$ and compute the dimensions of several homomorphism spaces involving the standard and projective modules. In Section 3.2, we develop category theory to prove Theorem 1.1. In Section 4, we construct the homomorphisms between adjacent standard modules by way of proving Theorem 1.2 and illustrate the irreducible representations of $\mathsf{TL}_n(0)$. In Section 5.1, we reformulate the notion of a Specht module in terms of polynomials. In Section 5.2, we work with (-1)-Specht modules using such a polynomial formulation over $\mathbb{F}_2$ and prove Theorem 1.3. In Section 6, we use the Ocneanu trace to prove Proposition 1.4. In Appendix A, we present elementary computations of several homomorphism spaces from Section 3.1 as well as a proof of the otherwise tedious Lemma 3.15 using composition series and diagram chasing.

2. Preliminaries

Let n be a positive integer.

Definition 2.1. *The Temperley-Lieb algebra $\mathsf{TL}_n(\beta)$ at some parameter $\beta \in \mathbb{C}$ is generated by the variables $e_1, e_2, \ldots, e_{n-1}$ on base field $\mathbb{C}$ under the presentation*

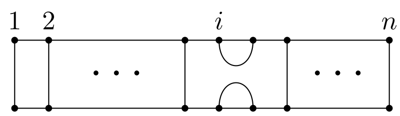
$$\mathsf{TL}_n(\beta) \cong \langle e_1, e_2, \ldots, e_{n-1} \mid e_i^2 = \beta e_i, e_i e_{i+1} e_i = e_i e_{i-1} e_i = e_i, e_i e_j = e_j e_i \quad \forall |i-j| \geq 2 \rangle.$$

Its dimension is the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Remark 2.2. *It may be helpful to formalize the notion of a diagram of strings. Specifically, every such diagram must consist of the following:*

- *a pair of horizontal lines,*
- *a collection of marked points on the horizontal lines, and,*
- *a collection of curves with endpoints being marked points such that no two curves intersect, and also that each marked point lies on exactly one curve.*

Then, each generator can be understood as a diagram of strings from n points above to n points below, such that

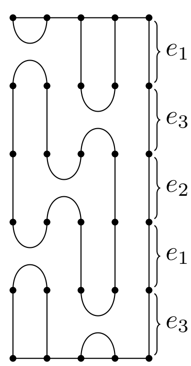
$e_i =$ 

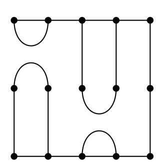
.

Multiplication of basis elements amounts to the concatenation of their respective diagrams, in which the bottom of the first diagram is identified with the top of the second. This preserves the algebraic relations given in Definition 2.1 as one may check [1]. In particular, all closed loops may be factored out as the scalar parameter β.

Definition 2.3. *Consider some diagram corresponding to a basis element in $\mathsf{TL}_n(\beta)$. We refer to curves connecting two points in its top row as cups and curves connecting two points in its bottom row as caps. Curves connecting the top and bottom rows are referred to as throughlines.*

Example 2.4. *Let us consider $e_1e_3e_2e_1e_3 \in \mathsf{TL}_5(\beta)$. We have*

$e_1e_3e_2e_1e_3 =$ 

$= \beta$  $= \beta e_1 e_3,$

as we factor out the closed loop in the string diagram for $e_1e_3e_2e_1e_3$.

Algebraically, we can use the braid relations to equivalently derive that

$$e_1e_3e_2e_1e_3 = e_1e_3e_2e_3e_1 = e_1e_3e_1 = e_1^2e_3 = \beta e_1e_3,$$

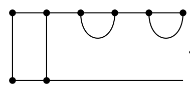
as expected.

Now let ℓ be some nonnegative integer such that $\ell \leq n$ and $\ell \equiv n \pmod 2$.

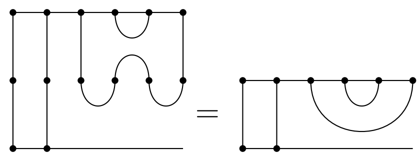
Definition 2.5. *Call a diagram of strings from* n *points above to* ℓ *points below monic if there are no caps. The* $\mathbb{C}$*-vector space spanned by the basis of all monic diagrams forms the standard module* W_ℓ^n*, which has dimension* $\binom{n}{\frac{n-\ell}{2}} - \binom{n}{\frac{n-\ell}{2}-1} = \frac{2\ell+2}{n+\ell+2}\binom{n}{\frac{n-\ell}{2}}$*.*

The standard modules are naturally acted upon by $\mathsf{TL}_n(\beta)$ *via concatenation of diagrams, with any resultant non-monic diagram due to the formation of caps defined to be equal to* 0*.*

Example 2.6. *Consider the basis element* $x \in W_2^6$ *given by*

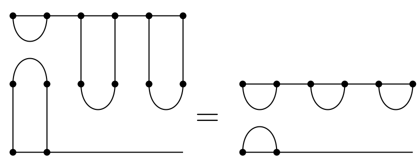
$x =$ .

We check that

$e_3 x =$ 

is monic and is thus another basis element of W_2^6*.*

Observe now that

$e_1 x =$ 

is not monic. In fact, the action of any basis element of $\mathsf{TL}_6(\beta)$ *on* $e_1 x$ *will result in a non-monic diagram. Hence* $e_1 x = 0$*.*

As discussed in Section 1, the standard modules are irreducible for generic values of β.

Definition 2.7. *The Hecke algebra* $\mathcal{H}_n(q)$ *at some parameter* $q \in \mathbb{C} \setminus \{0\}$ *is generated by the variables* $g_1, g_2, \ldots, g_{n-1}$ *on base field* $\mathbb{C}$ *under the presentation*

$$\mathcal{H}_q(n) \cong \langle g_1, g_2, \ldots, g_{n-1} \mid (g_i - q)(g_i + 1) = 0, g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, g_i g_j = g_j g_i \quad \forall |i-j| \geq 2 \rangle.$$

It has dimension $n!$*.*

Observe from Definitions 2.1 and 2.7 the similarities between the relations of the Temperley-Lieb and Hecke algebras. Indeed, as claimed in Section 1, there exists an epimorphism from $\mathcal{H}_n(q)$ to $\mathsf{TL}_n(\beta)$ when $\beta = q^{1/2} + q^{-1/2}$.

Proposition 2.8. *Let* $\beta, q \in \mathbb{C}$ *such that* $\beta = q^{1/2} + q^{-1/2}$ *and* $q \neq 0$*. Then there exists a homomorphism* $\theta \colon \mathcal{H}_n(q) \to \mathsf{TL}_n(\beta)$ *where* $\theta(g_i) = q^{1/2} e_i - 1$ *for all* i*. Moreover* θ *is surjective.*

Proof. We check that

$$\theta((g_i - q)(g_i + 1)) = (q^{1/2} e_i - 1 - q)(q^{1/2} e_i) = q e_i^2 - q(q^{1/2} + q^{-1/2}) e_i = q(e_i^2 - \beta e_i) = 0.$$

Thus θ respects the first relation of $\mathcal{H}_n(q)$.

We also have that

$$\begin{aligned}\theta(g_i g_{i+1} g_i) &= (q^{1/2}e_i - 1)(q^{1/2}e_{i+1} - 1)(q^{1/2}e_i - 1)\\ &= (q^{3/2} - q\beta + 2q^{1/2})e_i - q(e_ie_{i+1} + e_{i+1}e_i) + q^{1/2}e_{i+1} - 1\\ &= (q^{3/2} - q\beta + 2q^{1/2})e_{i+1} - q(e_ie_{i+1} + e_{i+1}e_i) + q^{1/2}e_i - 1\\ &= (q^{1/2}e_{i+1} - 1)(q^{1/2}e_i - 1)(q^{1/2}e_{i+1} - 1)\\ &= \theta(g_{i+1}g_ig_{i+1})\end{aligned}$$

and

$$\theta(g_ig_j) = (q^{1/2}e_i - 1)(q^{1/2}e_j - 1) = qe_ie_j - q^{1/2}(e_i + e_j) + 1 = (q^{1/2}e_j - 1)(q^{1/2}e_i - 1) = \theta(g_jg_i)$$

for $|i - j| \geq 2$, implying that θ respects the other two relations of $\mathcal{H}_n(q)$. Since $\theta(\frac{g_i+1}{q^{1/2}}) = e_i$, it follows that θ is an epimorphism. □

Definition 2.9. *Let n be a positive integer. The braid group B_n has presentation*

$$B_n \cong \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \quad \forall |i - j| \geq 2\rangle.$$

Each braid can be seen as n intersecting strands of string, in which each σ_i introduces a twist on the strands in the ith and $(i + 1)$th positions.

As one can check by comparing the relations of the presentations in Definitions 2.7 and 2.9, there is a natural homomorphism $\pi\colon B_n \to \mathcal{H}_n(q)$ given by $\pi(\sigma_i) = e_i$ for all i.

3. Proof of Theorem 1.1

3.1. **Homomorphism space computations.** For the next three sections, let us fix some positive even integer n and consider exclusively the specialization of the Temperley-Lieb algebra at $\beta = 0$. Since the W_ℓ^n are not necessarily irreducible at $\beta = 0$, we shall define a complete collection of irreducible modules of $\mathsf{TL}_n(0)$. To do so, we will also need to define a sequence of projective principle indecomposable modules, following the work of [28].

Definition 3.1. *Let $\ell > 0$. For any basis elements $x, y \in W_\ell^n$, let $\alpha(x)$ be the string diagram obtained by reflecting x horizontally. Obtain the element $\alpha(x, y) \in \mathsf{TL}_\ell(0)$ by diagrammatically concatenating $\alpha(x)$ above y. Now consider a pairing $\langle \cdot, \cdot \rangle$ on the basis elements of W_ℓ^n given by*

$$\langle x, y\rangle = \begin{cases} 1 & \text{if } \alpha(x, y) \in \mathsf{TL}_\ell(0) \text{ contains } \ell \text{ throughlines}\\ 0 & \text{otherwise.}\end{cases}$$

Extend the above into a bilinear pairing $\langle \cdot, \cdot \rangle\colon W_\ell^n \oplus W_\ell^n \to \mathbb{C}$. Then, define the quotient modules

$$L_\ell^n = W_\ell^n / \langle x \mid \langle x, y\rangle = 0 \quad \forall y \in W_\ell^n\rangle.$$

Definition 3.2. *Let $\{P_\ell^n \mid 2 \leq \ell \leq n, \ell \equiv 0 \pmod 2\}$ denote the collection of principal indecomposable modules, which are the indecomposable direct summands of the regular representation of $\mathsf{TL}_n(0)$. Each P_ℓ^n is projective.*

Observe that in Definition 3.2 we have chosen to index the principal indecomposable modules by positive even integers, which exactly matches the indexing of the quotient modules. This is well-defined as it is a general fact that the P_ℓ^n and the L_ℓ^n are in bijection with each other [8].

We will need the following results of [28].

Proposition 3.3 ([28], Corollary 4.2). *The isomorphism*

$$\mathrm{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} W^n_\ell \cong W^{n-1}_{\ell-1} \oplus W^{n-1}_{\ell+1}$$

holds.

Proposition 3.4 ([28], Corollary 7.4). *The collection* $\{L^n_\ell \mid 2 \le \ell \le n, \ell \equiv 0 \pmod 2\}$ *of quotient modules form a complete set of distinct irreducible modules of* $\mathsf{TL}_n(0)$*. Moreover, the sequence*

$$0 \longrightarrow L^n_{\ell+2} \longrightarrow W^n_\ell \longrightarrow L^n_\ell \longrightarrow 0$$

is exact and non-split for each ℓ*.*

Proposition 3.5 ([28], Proposition 8.2). *The following statements are true.*

(a) The indexing on the principle indecomposables may be chosen such that for each $\ell > 0$ *there exists an isomorphism*

$$P^n_\ell \cong \mathrm{Ind}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} W^{n-1}_{\ell-1}.$$

In particular, there exists a short exact sequence

$$0 \longrightarrow W^n_{\ell-2} \longrightarrow P^n_\ell \longrightarrow W^n_\ell \longrightarrow 0.$$

(b) The collection $\{W^{n-1}_{\ell-1} \mid 2 \le \ell \le n, \ell \equiv 0 \pmod 2\}$ *of standard modules form a complete set of pairwise distinct irreducible modules of* $\mathsf{TL}_{n-1}(0)$*. Additionally, the standard module* $W^{n-1}_{\ell-1}$ *is also projective and isomorphic to* $L^{n-1}_{\ell-1}$ *for each* $\ell > 0$*.*

Remark 3.6. *One may wonder whether there exists a diagrammatic depiction of the quotient and principle indecomposable modules. Indeed, from Proposition 3.5 we have the isomorphism* $P^n_\ell \cong \mathrm{Ind}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} W^{n-1}_{\ell-1}$*. The basis of the induced module* $\mathrm{Ind}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} W^{n-1}_{\ell-1}$ *can be interpreted as consisting of diagrams of strings from* n *points above to* ℓ *points below in which the only cap permitted connects the rightmost two points on the bottom.*

The diagrammatic construction of L^n_ℓ *is much more subtle. We shall give an explicit characterization of its structure with the proof of our second main result in Section 4.*

Now we deduce some novel results.

Corollary 3.7. *The algebra* $\mathsf{TL}_{n-1}(0)$ *is semisimple.*

Proof. Let V be some $\mathsf{TL}_{n-1}(0)$-module, and let $W \subseteq V$ be a submodule of maximal dimension. We have the short exact sequence

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.$$

Then V/W is irreducible and thus projective by maximality, implying that the exact sequence splits. Hence $V = (V/W) \oplus W$. We repeat this process on W by induction. As $\dim W < \dim V$, our process eventually terminates, expressing V as the direct sum of irreducibles. □

Proposition 3.8. *Let* ℓ *and* m *be positive even integers no greater than* n*. Then*

$$\dim \mathrm{Hom}(P^n_\ell, W^n_m) = \begin{cases} 1 & \text{if } \ell \in \{m, m+2\} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Frobenius reciprocity [14], we observe that

$$\operatorname{Hom}(P^n_\ell, W^n_m) = \operatorname{Hom}(\operatorname{Ind}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} W^{n-1}_{\ell-1}, W^n_m) = \operatorname{Hom}(W^{n-1}_{\ell-1}, \operatorname{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} W^n_m).$$

Hence by Proposition 3.3, we find that

$$\operatorname{Hom}(P^n_\ell, W^n_m) = \operatorname{Hom}(W^{n-1}_{\ell-1}, W^{n-1}_{m-1} \oplus W^{n-1}_{m+1}).$$

By Proposition 3.5, all three of $W^{n-1}_{\ell-1}$, W^{n-1}_{m-1}, and W^{n-1}_{m+1} are irreducible, so by Schur's Lemma

$$\dim \operatorname{Hom}(P^n_\ell, W^n_m) = |\{W^{n-1}_{\ell-1}\} \cap \{W^{n-1}_{m-1}, W^{n-1}_{m+1}\}|$$

and the result follows. ☐

Theorem 3.9. *Let ℓ and m be positive even integers no greater than n. Then*

$$\dim \operatorname{Hom}(P^n_\ell, P^n_m) = \begin{cases} 2 & \text{if } \ell = m \\ 1 & \text{if } |\ell - m| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

We will need the following lemma.

Lemma 3.10. *Let A be a $\mathbb{C}$-algebra, and let V, W, U_1, and U_2 be A-modules with $V \subseteq W$ and $W/V \cong U_1 \oplus U_2$. Then there exists some A-module V^* such that $V \subseteq V^* \subseteq W$, $V^*/V \cong U_1$, and $W/V^* \cong U_2$.*

Proof. Let $\pi\colon W \to U_1 \oplus U_2$ be the simple projection from W to $W/V = U_1 \oplus U_2$. We claim that $V^* = \pi^{-1}(U_1 \oplus \{0\})$ satisfies the desired properties. It is clear that $V^* \subseteq W$. We note also that the map sending each $w \in W$ to the projection of $\pi(w)$ onto U_2 is linear and has kernel V^*, establishing the isomorphism $W/V^* \cong U_2$.

Additionally, since $V = \ker \pi$, it follows that $V \subseteq V^*$. Then the map sending each $v \in V^*$ to the projection of $\pi(v)$ onto U_1 is linear and has kernel V, implying that $V^*/V \cong U_1$. ☐

Proof of Theorem 3.9. Note by Proposition 3.5 the short exact sequence

$$0 \longrightarrow W^n_{m-2} \longrightarrow P^n_m \longrightarrow W^n_m \longrightarrow 0.$$

Since exactness is retained upon restriction to $\mathsf{TL}_{n-1}(0)$, by Proposition 3.3 the sequence

$$0 \longrightarrow W^{n-1}_{m-3} \oplus W^{n-1}_{m-1} \longrightarrow \operatorname{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} P^n_m \longrightarrow W^{n-1}_{m-1} \oplus W^{n-1}_{m+1} \longrightarrow 0$$

is also exact. In particular, there exists a sequence of inclusions

$$0 \subseteq W^{n-1}_{m-3} \oplus W^{n-1}_{m-1} \subseteq \operatorname{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} P^n_m$$

with $\operatorname{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} P^n_m/(W^{n-1}_{m-3} \oplus W^{n-1}_{m-1}) \cong W^{n-1}_{m-1} \oplus W^{n-1}_{m+1}$. Then, by Lemma 3.10, there exists some $\mathsf{TL}_{n-1}(0)$-module V extending the sequence of inclusions to yield

$$0 \subseteq W^{n-1}_{m-3} \subseteq W^{n-1}_{m-3} \oplus W^{n-1}_{m-1} \subseteq V \subseteq \operatorname{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} P^n_m$$

where $V/(W^{n-1}_{m-3} \oplus W^{n-1}_{m-1}) \cong W^{n-1}_{m-1}$ and $\operatorname{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} P^n_m/V \cong W^{n-1}_{m+1}$. Since W^{n-1}_{m-3}, W^{n-1}_{m-1}, and W^{n-1}_{m+1} are all irreducible by Proposition 3.5, it follows that the above inclusions form a composition series for $\operatorname{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} P^n_m$, yielding the multiset $\{W^{n-1}_{m-3}, W^{n-1}_{m-1}, W^{n-1}_{m-1}, W^{n-1}_{m+1}\}$ of subquotients.

By Frobenius reciprocity and Proposition 3.5, we have

$$\mathrm{Hom}(P^n_\ell, P^n_m) = \mathrm{Hom}(\mathrm{Ind}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} W^{n-1}_{\ell-1}, P^n_m) = \mathrm{Hom}(W^{n-1}_{\ell-1}, \mathrm{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} P^n_m).$$

Then, Schur's Lemma implies that the dimension of this homomorphism space is exactly the multiplicity of $W^{n-1}_{\ell-1}$ in the multiset of subquotients of $\mathrm{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} P^n_m$. As a result

$$\begin{aligned} \dim \mathrm{Hom}(P^n_\ell, P^n_m) &= \dim \mathrm{Hom}(W^{n-1}_{\ell-1}, \mathrm{Res}^{\mathsf{TL}_n(0)}_{\mathsf{TL}_{n-1}(0)} P^n_m) \\ &= |\{W^{n-1}_{\ell-1}\} \cap \{W^{n-1}_{m-3}, W^{n-1}_{m-1}, W^{n-1}_{m-1}, W^{n-1}_{m+1}\}|. \end{aligned}$$

The desired conclusion now follows. □

The reader unfamiliar with the concept of subquotients is advised to refer to the results developed in Appendix A.

3.2. **Category equivalence.** We now introduce the concept of quivers and path algebras [7] in order to formulate the necessary category equivalence.

Definition 3.11. *A quiver $\mathcal{Q}$ is a directed graph in which loops and multiple edges are allowed. A path in $\mathcal{Q}$ is defined in the familiar graph-theoretic manner, in which vertices and edges are permitted to appear multiple times. Trivial paths, which start and end at the same vertex and contain no edges, are also allowed. For every path p of $\mathcal{Q}$, let $s(p)$ and $t(p)$ respectively denote the starting and terminal vertices of p.*

Given two paths p and q of $\mathcal{Q}$ such that $s(p) = t(q)$, we let $p \circ q$ be the path that starts at $s(q)$, traverses along q to reach $s(p) = t(q)$, and then traverses along p to terminate at $t(p)$.

Definition 3.12. *Let $\mathcal{Q}$ be a quiver. The path algebra $\mathbb{C}\mathcal{Q}$ of $\mathcal{Q}$ is the $\mathbb{C}$-vector space spanned by all paths on $\mathcal{Q}$ such that, for paths p and q of $\mathcal{Q}$, we have*

$$pq = \begin{cases} p \circ q & \text{if } s(p) = t(q) \\ 0 & \text{otherwise.} \end{cases}$$

A representation of $\mathcal{Q}$ is a collection of vector spaces and maps endowed with a bijection assigning each vertex of $\mathcal{Q}$ to a vector space and each directed edge e of $\mathcal{Q}$ to a map between the vector spaces associated with $s(e)$ and $t(e)$. It is well-known that the representations of $\mathcal{Q}$ are in bijection with the $\mathbb{C}\mathcal{Q}$-modules. Hence, the category of representations of $\mathcal{Q}$, denoted $\mathbf{Rep}(\mathbb{C}\mathcal{Q})$*, is exactly the category of $\mathbb{C}\mathcal{Q}$-modules.*

Let us now define the straight-line quiver for some positive integer n.

Definition 3.13. *The straight-line quiver $\mathcal{Q}_n$ is the quiver on n vertices has the structure*

$$\bullet \underset{b_1}{\overset{a_1}{\rightleftarrows}} \bullet \underset{b_2}{\overset{a_2}{\rightleftarrows}} \bullet \underset{b_3}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-1}}{\rightleftarrows}} \bullet.$$

We let e_i denote the trivial path on the ith leftmost vertex.

We now give the full statement of Theorem 1.1 and establish its proof.

Theorem 3.14 (Theorem 1.1)**.** *In the path algebra $\mathbb{C}\mathcal{Q}_{n/2}$, define the ideal*

$$J = \langle a_{i+1}a_i, b_ib_{i+1}, a_ib_i - b_{i+1}a_{i+1} \mid 0 \le i \le n-2\rangle.$$

Then the functor $\mathbf{\Phi}\colon \mathbf{Rep}(\mathsf{TL}_n(0)) \to \mathbf{Rep}(\mathbb{C}\mathcal{Q}_{n/2}/J)$ such that

$$\mathbf{\Phi}(X) = \operatorname{Hom}(P_2^n, X) \rightleftarrows \operatorname{Hom}(P_4^n, X) \rightleftarrows \operatorname{Hom}(P_6^n, X) \rightleftarrows \cdots \rightleftarrows \operatorname{Hom}(P_n^n, X)$$

establishes an equivalence $\mathbf{Rep}(\mathsf{TL}_n(0)) \simeq \mathbf{Rep}(\mathbb{C}\mathcal{Q}_{n/2}/J)$.

Proof. Note that $P^{\circ n} = \bigoplus_{i=1}^{n/2} P_{2i}^n$ is a projective of $\mathbf{Rep}(\mathsf{TL}_n(0))$ as projectivity is preserved under direct sums. By Propositions 3.4 and 3.5 the collection of irreducible subquotients of P_ℓ^n include $L_{\ell+c}^n$ for all $c \in \{-2, 0, 2\}$, so every irreducible object of $\mathbf{Rep}(\mathsf{TL}_n(0))$ is included as a subquotient within the composition series of $P^{\circ n}$. Hence for any object $X \in \mathbf{Rep}(\mathsf{TL}_n(0))$ we have $\operatorname{Hom}(P^{\circ n}, X) \neq 0$. Thus $P^{\circ n}$ is a projective generator of $\mathbf{Rep}(\mathsf{TL}_n(0))$, and the Gabriel-Popesco Theorem [26] implies that the functor sending X to $\operatorname{Hom}(P^{\circ n}, X) \in \mathbf{Rep}(\operatorname{End}(P^{\circ n}))$ establishes a category equivalence. Hence $\mathbf{Rep}(\mathsf{TL}_n(0)) \simeq \mathbf{Rep}(\operatorname{End}(P^{\circ n}))$.

Now let $\omega_\ell^n\colon P_\ell^n \to P_{\ell+2}^n$ and $\gamma_\ell^n\colon P_{\ell+2}^n \to P_\ell^n$ be nonzero maps between adjacent projectives. By Theorem 3.9, they are fixed up to a constant and over the projectives they attain the structure

$$P_2^n \underset{\gamma_2^n}{\overset{\omega_2^n}{\rightleftarrows}} P_4^n \underset{\gamma_4^n}{\overset{\omega_4^n}{\rightleftarrows}} P_6^n \underset{\gamma_6^n}{\overset{\omega_6^n}{\rightleftarrows}} \cdots \underset{\gamma_{n-2}^n}{\overset{\omega_{n-2}^n}{\rightleftarrows}} P_n^n.$$

We make use of the following lemma, whose proof we defer to Appendix A.

Lemma 3.15. *For every $\ell < n$, the compositions $\gamma_\ell^n \circ \omega_\ell^n$ and $\omega_\ell^n \circ \gamma_\ell^n$ are nonzero and nonidentity.*

Let $\pi_\ell^n\colon P^{\circ n} \to P_\ell^n$ and $\iota_\ell^n\colon P_\ell^n \to P^{\circ n}$ be the standard projection and inclusion maps. We consider the homomorphism of algebras $\Psi\colon \mathbb{C}\mathcal{Q}_{n/2}/J \to \operatorname{End}(P^{\circ n})$ where $\Psi(a_i) = \iota_{2i+2}^n \circ \omega_{2i}^n \circ \pi_{2i}^n$, $\Psi(b_i) = \iota_{2i}^n \circ \gamma_{2i}^n \circ \pi_{2i+2}^n$, and $\Psi(e_i) = \pi_{2i}^n$. We observe that

$$\Psi(a_{i+1}a_i) = (\iota_{2i+4}^n \circ \omega_{2i+2}^n \circ \pi_{2i+2}^n) \circ (\iota_{2i+2}^n \circ \omega_{2i}^n \circ \pi_{2i}^n) = \iota_{2i+4}^n \circ \omega_{2i+2}^n \circ \omega_{2i}^n \circ \pi_{2i}^n = 0$$

as $\omega_{2i+2}^n \circ \omega_{2i}^n \in \operatorname{Hom}(P_{2i}^n, P_{2i+4}^n) = \{0\}$ by Theorem 3.9. Similarly, we have

$$\Psi(b_ib_{i+1}) = (\iota_{2i}^n \circ \gamma_{2i}^n \circ \pi_{2i+2}^n) \circ (\iota_{2i+2}^n \circ \gamma_{2i+2}^n \circ \pi_{2i+4}^n) = \iota_{2i}^n \circ \gamma_{2i}^n \circ \gamma_{2i+2}^n \circ \pi_{2i+4}^n = 0$$

since $\gamma_{2i}^n \circ \gamma_{2i+2}^n \in \operatorname{Hom}(P_{2i+4}, P_i) = \{0\}$ again by Theorem 3.9. Finally, by Lemma 3.15, we note that $\omega_\ell^n \circ \gamma_\ell^n$ and $\gamma_{\ell+2}^n \circ \omega_{\ell+2}^n$ are both nonzero and nonidentity endomorphisms of $P_{\ell+2}^n$. As $\dim \operatorname{End}(P_{\ell+2}^n) = 2$ by Theorem 3.9, it follows that $\omega_\ell^n \circ \gamma_\ell^n$ and $\gamma_{\ell+2}^n \circ \omega_{\ell+2}^n$ are the same up to a constant multiplier, so without loss of generality we may assume that $\omega_\ell^n \circ \gamma_\ell^n = \gamma_{\ell+2}^n \circ \omega_{\ell+2}^n$. Thus

$$\Psi(a_ib_i) = \iota_{2i+2}^n \circ \omega_{2i}^n \circ \gamma_{2i}^n \circ \pi_{2i+2}^n = \iota_{2i+2}^n \circ \gamma_{2i+2}^n \circ \omega_{2i+2}^n \circ \pi_{2i+2}^n = \Psi(b_{i+1}a_{i+1}),$$

so it follows that $J \in \ker \Psi$, implying that Ψ is a well-defined homomorphism.

Next we observe the surjectivity of Ψ. Note that $\operatorname{End}(P^{\circ n}) \cong \bigoplus_{1 \le i,j \le n/2} \operatorname{Hom}(P_{2i}, P_{2j})$. By Theorem 3.9, the one-dimensional space $\operatorname{Hom}(P_{2i}, P_{2i+2})$ is spanned by ω_{2i}^n, which is exactly the image of a_i, disregarding inclusions and projections. Similarly, the space $\operatorname{Hom}(P_{2i+2}, P_{2i})$ is one-dimensional and spanned by the image of b_i, disregarding inclusions and projections. The space $\operatorname{End}(P_{2i})$ is two-dimensional and spanned by the images of e_i and b_ia_i. All other summands of $\bigoplus_{1 \le i,j \le n/2} \operatorname{Hom}(P_{2i}, P_{2j})$ vanish, so surjectivity is immediate.

Finally, we check that $\dim \mathbb{C}\mathcal{Q}_{n/2}/J$ can be obtained by summing over the number of distinct paths from the ith to jth leftmost vertices of $\mathcal{Q}_{n/2}$ up to quotienting by J. Let this number be $\xi(i,j)$. First note that if $|i-j| \geq 2$ then any path between the vertices in question have a subpath given by $a_{k+1}a_k = 0$ or $b_k b_{k+1} = 0$, so $\xi(i,j) = 0$. Next, if $|i-j| = 1$, then $\xi(i,j) = 1$ given by the obvious path of length 1, as otherwise we are forced to have a subpath of $a_{k+1}a_k = 0$ or $b_k b_{k+1} = 0$. For $i = j$, we have $\xi(i,i) \geq 2$ as e_i and $b_i a_i$ are both valid. All other paths vanish via quotienting by $a_{k+1}a_k$ or $b_k b_{k+1}$, or fall to $b_i a_i$ via quotienting by $a_k b_k - b_{k+1}a_{k+1}$. Thus $\xi(i,i) = 2$. It follows that $\xi(i,j) = \dim \operatorname{Hom}(P_{2i}, P_{2j})$, and hence

$$\dim \operatorname{End}(P^{\circ n}) = \sum_{1 \leq i,j \leq n/2} \dim \operatorname{Hom}(P_{2i}, P_{2j}) = \sum_{1 \leq i,j \leq n/2} \xi(i,j) = \dim \mathbb{C}\mathcal{Q}_{n/2}/J.$$

As Ψ is an epimorphism on algebras of equal dimension, it follows that Ψ is an isomorphism. Thus $\operatorname{End}(P^{\circ n}) \cong \mathbb{C}\mathcal{Q}_{n/2}/J$, and $\mathbf{Rep}(\mathsf{TL}_n(0)) \cong \mathbf{Rep}(\mathbb{C}\mathcal{Q}_{n/2}/J)$ as claimed.

We may compose the functor mapping $X \in \mathbf{Rep}(\mathsf{TL}_n(0))$ to $\operatorname{Hom}(P^{\circ n}, X) \in \mathbf{Rep}(\operatorname{End}(P^{\circ n}))$ with the inverse of Ψ, in which an endomorphism of $P^{\circ n}$ is sent to a path on $\mathcal{Q}_{n/2}$. Applying the Gabriel-Popesco Theorem once more, this yields a functor $\mathbf{\Phi}\colon \mathbf{Rep}(\mathsf{TL}_n(0)) \to \mathbf{Rep}(\mathbb{C}\mathcal{Q}_{n/2}/J)$ mapping X to the object

$$\operatorname{Hom}(P_2^n, X) \rightleftarrows \operatorname{Hom}(P_4^n, X) \rightleftarrows \operatorname{Hom}(P_6^n, X) \rightleftarrows \cdots \rightleftarrows \operatorname{Hom}(P_n^n, X),$$

which we interpret as a representation of $\mathcal{Q}_{n/2}$ by Definition 3.12. Since the P_ℓ^n are all projective, the functor $\mathbf{\Phi}$ is exact, preserving the exactness of sequences. ☐

Corollary 3.16. *There exists a long exact sequence on the standard modules of the form*

$$0 \longrightarrow W_n^n \longrightarrow W_{n-2}^n \longrightarrow \cdots \longrightarrow W_2^n \longrightarrow W_0^n \longrightarrow 0.$$

Proof. Retain the same notation from Theorem 3.14. By Proposition 3.8, we have

$$\mathbf{\Phi}(W_n^n) = \ 0 \rightleftarrows 0 \rightleftarrows 0 \rightleftarrows \cdots \rightleftarrows \mathbb{C},$$

where the last entry of $\mathbf{\Phi}(W_n^n)$ is $\mathbb{C}$ and everything else vanishes.

If $2 \leq \ell \leq n-2$ and ℓ is even, we check by Proposition 3.8 that

$$\mathbf{\Phi}(W_\ell^n) = \ 0 \rightleftarrows 0 \rightleftarrows \cdots \rightleftarrows \mathbb{C} \rightleftarrows \mathbb{C} \rightleftarrows \cdots \rightleftarrows 0,$$

where only the entries corresponding to $\operatorname{Hom}(P_\ell^n, W_\ell^n)$ and $\operatorname{Hom}(P_{\ell+2}^n, W_\ell^n)$ are $\mathbb{C}$.

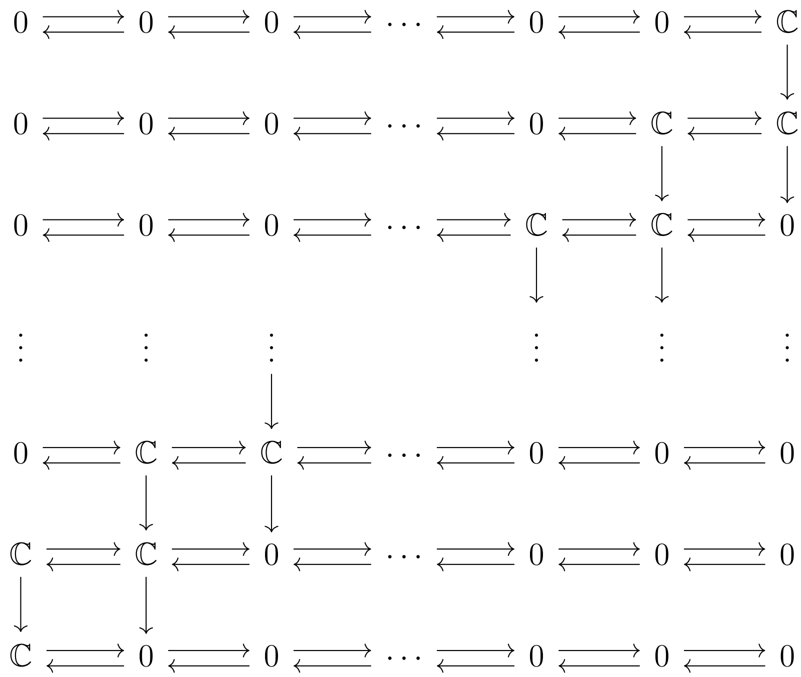
Finally, we again have by Proposition 3.8 that

$$\mathbf{\Phi}(W_0^n) = \ \mathbb{C} \rightleftarrows 0 \rightleftarrows 0 \rightleftarrows \cdots \rightleftarrows 0,$$

where the first entry of $\mathbf{\Phi}(W_0^n)$ is $\mathbb{C}$ and everything else is zero.

In particular, by the exactness of $\mathbf{\Phi}$ and the category equivalence established in Theorem 3.14, the claimed long exact sequence exists if and only if there exists a long exact sequence on

Rep$(\mathbb{C}\mathcal{Q}_{n/2}/J)$ of the form



but this is immediate if we let all leftward-pointing endomorphisms of $\mathbb{C}$ be the zero map and let all other endomorphisms of $\mathbb{C}$ be the identity map. □

Remark 3.17. *The observant reader may notice the resemblance between the quotient* $\mathbb{C}\mathcal{Q}_n/J$ *and the category of perverse sheaves on* $\mathbb{P}^n$. *In particular, consider the category* $\mathbf{Perv}(\mathbb{P}^n)$ *of perverse sheaves on* $\mathbb{P}^n$ *with the stratification* $\mathbb{P}^n = \bigcup_{i=1}^n \mathbb{A}^i \cup \{\mathrm{pt}\}$ [4]. *It is well known that the* $\mathbf{Perv}(\mathbb{P}^n)$ *is equivalent to the category* $\mathbf{Rep}(\mathbb{C}\mathcal{Q}_n/J^*)$ *where*

$$J^* = \langle a_{i+1}a_i, b_ib_{i+1}, a_ib_i - b_{i+1}a_{i+1}, b_1a_1 \mid 0 \leq i \leq n-2\rangle,$$

consisting of J *with the added relation that* $b_1a_1 = 0$ [6].

Since by Lemma 3.15 the maps $\gamma_2^n \circ \omega_2^n$ *and* $\omega_{n-2}^n \circ \gamma_{n-2}^n$ *are both nonzero, there is no isomorphism of algebras between* $\mathrm{End}(P^{\circ n})$ *and* $\mathbb{C}\mathcal{Q}_{n/2}/J^*$. *It is necessary to remove the relation* b_1a_1 *to obtain the ideal* J *as in Theorem 3.14. Alternatively, we can view* $\mathbf{Rep}(\mathbb{C}\mathcal{Q}_n/J)$ *as being* $\mathbf{Perv}(\mathbb{P}^{n+1})$ *with the leftmost vertex of its corresponding quiver removed.*

4. Proof of Theorem 1.2

In this section, we shall construct an explicit sequence of homomorphisms between adjacent standard modules that give rise to the long exact sequence implied by Corollary 3.16.

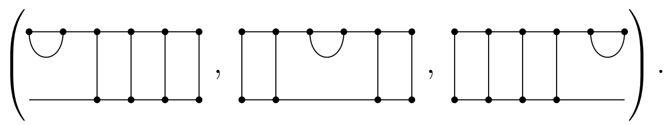
Definition 4.1. *For some nonnegative positive integer* $i \leq \ell$, *let* $\delta_i^\ell \in W_\ell^{\ell+2}$ *be the basis element consisting of a cup surrounded by* i *throughlines on the left and* $\ell - i$ *throughlines on the right. Observe that there is a right action of* $W_\ell^{\ell+2}$ *on* $W_{\ell+2}^n$ *naturally given by concatenating diagrams. For some basis element* $x \in W_{\ell+2}^n$, *the diagram* $x\delta_i^\ell$ *connects the* $(i+1)$*th and* $(i+2)$*th lower leftmost points of* x *with a cup.*

Let $\phi_\ell^n\colon W_{\ell+2}^n \to W_\ell^n$ be the linear map given by the right action

$$\phi_\ell^n(x) = x\sum_{i=0}^{\ell/2}(-1)^i\delta_{2i}^n.$$

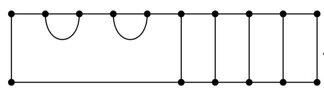
Example 4.2. *In W_4^6, we may observe for instance the elements*

$$(\delta_0^4, \delta_2^4, \delta_4^4) =$$



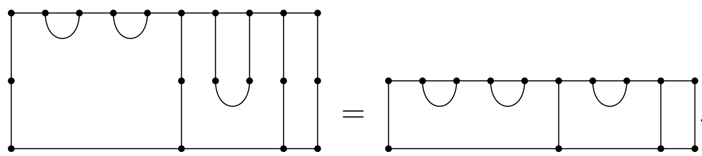
Consider the element $x \in W_6^{10}$ given by

$x =$



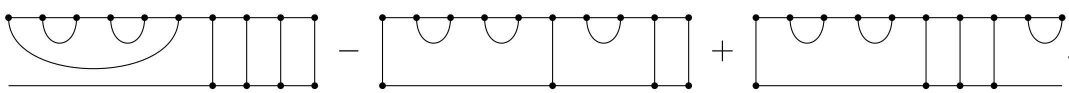
The diagram $x\delta_2^4$ entails joining the third and fourth lower leftmost points of x, yielding

$x\delta_2^4 =$



In particular, we have $\phi_4^{10}(x) = x(\delta_0^4 - \delta_2^4 + \delta_4^4)$, so that

$\phi_4^{10}(x) =$

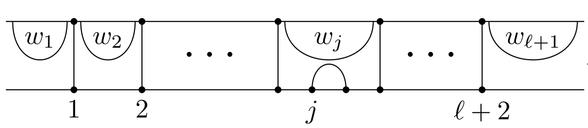


Proposition 4.3. *The map ϕ_ℓ^n is a well-defined homomorphism of standard modules.*

Proof. We first show that $\phi_\ell^n(x) = 0$ as long as $x = 0$. Equivalently, we may suppose that x is a non-monic string diagram. Let k be the number of caps in x. Observe that a cup is created if $x\delta_i^\ell$ joins two throughlines of x; otherwise, if $x\delta_i^\ell$ joins a cap with a throughline or another cap, then exactly one cap is removed. Hence, the diagram $x\delta_i^\ell$ has at most $k-1$ caps. As a result, if $k \geq 2$, then $\phi_\ell^n(x) = 0$.

Now suppose that $k = 1$. Then x is of the form

$x =$



where the sole cap connects the jth and $(j+1)$th leftmost points on the bottom and the w_i are subdiagrams consisting only of nested cups. We now consider the parity of j.

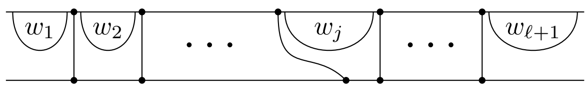
If j is odd, then in order for $x\delta_i^\ell$ to feature no caps while i is even we must have $i = j-1$. Hence $x\delta_i^\ell = 0$ for all even i such that $i \neq j-1$. However, observe that $x\delta_{j-1}^\ell$ is formed by joining the jth and $(j+1)$th leftmost points on the bottom, thus completing a closed loop and vanishing due to the specialization $\beta = 0$. Since $\phi_\ell^n(x)$ is a linear combination of the $x\delta_i^\ell$ restricted to even values of i, it follows that $\phi_\ell^n(x) = 0$.

Otherwise, if j is even, then for $x\delta_i^\ell$ to have no caps while i is even we must have $i \in \{j-2, j\}$. Hence, we have $x\delta_i^\ell = 0$ for all even $i \notin \{j-2, j\}$, so it follows that

$$\phi_\ell^n(x) = (-1)^{j/2-1} x(\delta_{j-2}^\ell - \delta_j^\ell).$$

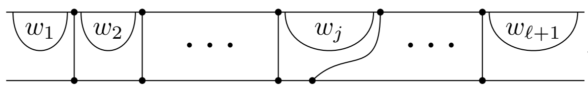
However, note that

$$x\delta_{j-2}^\ell =$$



and

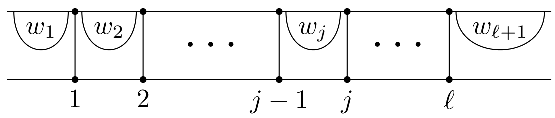
$$x\delta_j^\ell =$$



.

As a result

$$x\delta_{j-2}^\ell =$$



$$= x\delta_j^\ell$$

and thus $\phi_\ell^n(x) = 0$. Thus, in both cases, we have $\phi_\ell^n(x) = 0$, so ϕ_ℓ^n is indeed well-defined.

Now it suffices to verify that ϕ_ℓ^n intertwines. But this is apparent as the left action of $\mathsf{TL}_n(0)$ on W_ℓ^n operates by concatenation above, while ϕ_ℓ^n acts by concatenation below. □

Remark 4.4. *Observe that the alternating sum in the construction of ϕ_ℓ^n is necessary as in the proof of Proposition 4.3 it causes the two equivalent string diagrams to cancel to zero in the case where j is even. Indeed, Proposition 4.3 would fail to hold if ϕ_ℓ^n was defined to be a direct or incomplete summation of the $x\delta_i^\ell$.*

Additionally, the reason for the specialization $\beta = 0$ becomes clear in the case where j is odd, as a closed loop is created while everything else is canceled out. In order for this expression to be zero, we must force the specialization $\beta = 0$. This implies that the standard modules under this specialization are not themselves irreducibles, which makes sense as a long exact sequence can never arise in a collection of irreducible representations due to Schur's Lemma.

Proposition 4.5. *The composition $\phi_{\ell-2}^n \circ \phi_\ell^n = 0$ holds.*

Proof. It suffices to show that $\phi_{\ell-2}^n(\phi_\ell^n(x)) = 0$ for all basis elements x.

Observe that

$$\begin{aligned}
\phi_{\ell-2}^n(\phi_\ell^n(x)) &= \phi_{\ell-2}^n\left(x\sum_{i=0}^{\ell/2}(-1)^i\delta_{2i}^\ell\right) \\
&= \left(x\sum_{i=0}^{\ell/2}(-1)^i\delta_{2i}^\ell\right)\left(\sum_{j=0}^{\ell/2-1}(-1)^j\delta_{2j}^{\ell-2}\right) \\
&= x\sum_{i=0}^{\ell/2}\sum_{j=0}^{\ell/2-1}(-1)^{i+j}\delta_{2i}^\ell\delta_{2j}^{\ell-2}.
\end{aligned}$$

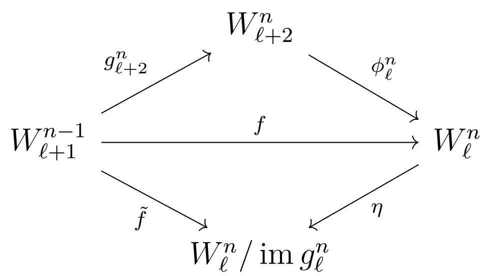
Note for $i > j$ that $\delta_{2i}^\ell\delta_{2j}^{\ell-2} = \delta_{2j}^\ell\delta_{2i-2}^{\ell-2}$, while for $i \le j$ we have $\delta_{2i}^\ell\delta_{2j}^{\ell-2} = \delta_{2j+2}^\ell\delta_{2i}^{\ell-2}$. In both cases, we end up joining the same pairs of points on the bottom edge of the diagram. Hence by a pairing argument the sign factor $(-1)^{i+j}$ causes the claimed double summation to vanish. □

Now we are ready to state our main result, which proves a major portion of Theorem 1.2.

Theorem 4.6 (Theorem 1.2)**.** *The standard modules W_ℓ^n and the maps given by ϕ_ℓ^n constitute a long exact sequence of homomorphisms given by*

$$0 \xrightarrow{\phi_n^n} W_n^n \xrightarrow{\phi_{n-2}^n} W_{n-2}^n \xrightarrow{\phi_{n-4}^n} \cdots \xrightarrow{\phi_2^n} W_2^n \xrightarrow{\phi_0^n} W_0^n \longrightarrow 0.$$

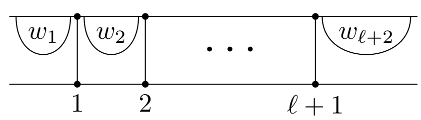
To prove Theorem 4.6, let $g_{\ell+2}^n \colon W_{\ell+1}^{n-1} \to W_{\ell+2}^n$ be the linear map such that $g_{\ell+2}^n(x)$ attaches a rightmost throughline to the basis element x. Let $f = \phi_\ell^n \circ g_{\ell+2}^n$. Finally, let $\eta \colon W_\ell^n \to W_\ell^n / \operatorname{im} g_\ell^n$ be the simple projection and set $\tilde{f} = \eta \circ f$. These maps commute as follows.



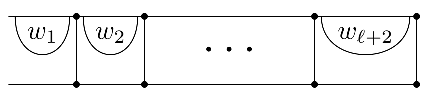
We first prove the following lemma.

Lemma 4.7. *The map $\tilde{f}$ is a bijection of vector spaces between $W_{\ell+1}^{n-1}$ and $W_\ell^n / \operatorname{im} g_\ell^n$.*

Proof. Observe that $\tilde{f} \colon W_{\ell+1}^{n-1} \to W_\ell^n / \operatorname{im} g_\ell^n$ satisfies $\tilde{f} = \eta \circ \phi_\ell^n \circ g_{\ell+2}^n$. Thus, letting $x \in W_{\ell+1}^{n-1}$ be a generator, we can characterize it as

$x =$ 

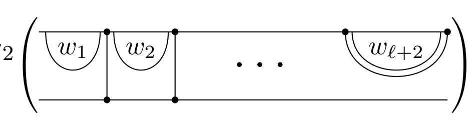
where the w_i are subdiagrams consisting only of cups. Thus

$g_{\ell+2}^n(x) =$ 

We have

$$\phi_\ell^n(g_{\ell+2}^n(x)) = \sum_{i=0}^{\ell/2} (-1)^i g_{\ell+2}^n(x) \delta_{2i}^\ell.$$

Based on the above depiction of $g_{\ell+2}^n(x)$, we observe that $g_{\ell+2}^n(x)\delta_{2i}^\ell$ will always have a rightmost throughline unless $i = \frac{\ell}{2}$, in which we connect the two rightmost points on the bottom of $g_{\ell+2}^n(x)$. It follows that η annhiliates all elements in the above summation except for $g_{\ell+2}^n(x)\delta_\ell^\ell$. Hence

$\tilde{f}(x) = \eta(\phi_\ell^n(g_{\ell+2}^n(x))) = (-1)^{\ell/2} g_{\ell+2}^n(x)\delta_\ell^\ell = (-1)^{\ell/2}$ .

In other words, the map $\tilde{f}$ simply bends the rightmost throughline of some $x \in W_{\ell+1}^{n-1}$ into the rightmost maximal arc of x while leaving everything else intact. The bijective nature of $\tilde{f}$ is thus apparent, as desired. ∎

Proof of Theorem 4.6. Let us retain the notation used in Lemma 4.7. Since $f = \phi_\ell^n \circ g_{\ell+2}^n$, we have $\operatorname{im} f \subseteq \operatorname{im} \phi_\ell^n$. Additionally, the bijective nature of $\tilde{f}$ implies that f is injective. We thus have the inequality chain

$$\dim \operatorname{im} \phi_\ell^n \geq \dim f \geq \dim W_{\ell+1}^{n-1},$$

implying that

$$\dim \operatorname{im} \phi_\ell^n \geq \dim W_{\ell+1}^{n-1},$$

and, shifting indices,

$$\dim \operatorname{im} \phi_{\ell-2}^n \geq \dim W_{\ell-1}^{n-1}.$$

On the other hand, by Proposition 4.5 we have $\phi_{\ell-2}^n \circ \phi_\ell^n = 0$ for all ℓ, so $\operatorname{im} \phi_\ell^n \subseteq \ker \phi_{\ell-2}^n$. In particular, rank-nullity implies that

$$\dim \operatorname{im} \phi_{\ell-2}^n + \dim \operatorname{im} \phi_\ell^n \leq \dim \operatorname{im} \phi_{\ell-2}^n + \dim \ker \phi_{\ell-2}^n = \dim W_\ell^n.$$

Now note that

$$\begin{aligned}\dim W_\ell^n &= \binom{n}{\frac{n-\ell}{2}} - \binom{n}{\frac{n-\ell}{2} - 1} \\ &= \left(\binom{n-1}{\frac{n-\ell}{2}} - \binom{n-1}{\frac{n-\ell}{2} - 1}\right) + \left(\binom{n-1}{\frac{n-\ell}{2} - 1} - \binom{n-1}{\frac{n-\ell}{2} - 2}\right) \\ &= \dim W_{\ell-1}^{n-1} + \dim W_{\ell+1}^{n-1}\end{aligned}$$

either due to the above computation or by Proposition 3.3. Either way, this forces the equality cases in which $\dim \operatorname{im} \phi_{\ell-2}^n = \dim W_{\ell-1}^{n-1}$ and $\dim \operatorname{im} \phi_\ell^n = \dim W_{\ell+1}^{n-1}$.

Since we have shown previously that $\operatorname{im} \phi_\ell^n \subseteq \ker \phi_{\ell-2}^n$, the result follows. $\square$

The final portion of Theorem 1.2 can now be proven quite easily.

Corollary 4.8 (Theorem 1.2)**.** *For all $\ell \leq n-2$, the isomorphism $\operatorname{im} \phi_\ell^n \cong L_{\ell+2}^n$ holds, In particular, the collection $\{\operatorname{im} \phi_\ell^n \mid 0 \leq \ell \leq n-2, \ell \equiv 0 \pmod 2\}$ forms a complete set of distinct irreducible modules of $\mathsf{TL}_n(0)$.*

Proof. Since $\phi_{\ell-2}^n \colon W_\ell^n \to W_{\ell-2}^n$ is a valid homomorphism, it is automatic that the sequence

$$0 \longrightarrow \ker \phi_{\ell-2}^n \longrightarrow W_\ell^n \xrightarrow{\phi_{\ell-2}^n} \operatorname{im} \phi_{\ell-2}^n \longrightarrow 0$$

is exact. By Theorem 4.6, we have $\operatorname{im} \phi_\ell^n = \ker \phi_{\ell-2}^n$ and so the short exact sequence becomes

$$0 \longrightarrow \operatorname{im} \phi_\ell^n \longrightarrow W_\ell^n \xrightarrow{\phi_{\ell-2}^n} \operatorname{im} \phi_{\ell-2}^n \longrightarrow 0.$$

Recall by Proposition 3.4 that each standard module has a composition series containing exactly two irreducible quotients, namely due to the short exact sequence

$$0 \longrightarrow L_{\ell+2}^n \longrightarrow W_\ell^n \longrightarrow L_\ell^n \longrightarrow 0.$$

As a result, we have $\operatorname{im} \phi_\ell^n \cong L_{\ell+2}^n$ for all $\ell \leq n-2$. $\square$

Remark 4.9. *Note that Corollary 4.8 implies the irreducibility of the first and last standard modules as $W_0^n \cong \operatorname{im} \phi_0^n \cong L_2^n$ and $W_n^n \cong \operatorname{im} \phi_{n-2}^n \cong L_n^n$.*

5. Proof of Theorem 1.3

5.1. **Specht polynomials.** Let n be a positive integer, and fix a field $\mathbb{k}$ of characteristic zero. We now turn our attention to the Specht modules over $\mathbb{k}$.

Definition 5.1. *Let $\lambda \vdash n$ be a partition of n, and let t^λ be its canonical Young tableau. Let the row stabilizer $P_\lambda \subseteq \mathfrak{S}_n$ be the subgroup containing all permutations σ for which i and $\sigma(i)$ are cell labels appearing in the same row of t^λ for all i. Define the column stabilizer Q_λ analogously, this time containing all σ for which i and $\sigma(i)$ share a column in t^λ for all i.*

Definition 5.2. *The Young symmetrizer $c_\lambda \in \mathbb{k}[\mathfrak{S}_n]$ and dual Young symmetrizer $c_\lambda^* \in \mathbb{k}[\mathfrak{S}_n]$ are given by*

$$c_\lambda = \left(\sum_{\sigma \in P_\lambda} e_\sigma\right)\left(\sum_{\sigma \in Q_\lambda} \operatorname{sgn}(\sigma) e_\sigma\right)$$

and

$$c_\lambda^* = \left(\sum_{\sigma \in P_{\lambda^\top}} \operatorname{sgn}(\sigma) e_\sigma\right)\left(\sum_{\sigma \in Q_{\lambda^\top}} e_\sigma\right).$$

The Specht module S^λ is the right ideal $\mathbb{k}[\mathfrak{S}_n]c_\lambda$ under the permutation action of $\mathbb{k}[\mathfrak{S}_n]$. They constitute the irreducible representations of $\mathbb{k}[\mathfrak{S}_n]$ for $\mathbb{k}$ of characteristic zero.

Example 5.3. *Consider the partition*

$$\lambda = (3,1) = \begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

for $n = 4$. It has canonical Young tableaux

$$t^\lambda = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

Then $P_\lambda = \langle (1\,2), (1\,3)\rangle$ and $Q_\lambda = \langle (1\,4)\rangle$. Hence, we have

$$c_\lambda = (e_1 + e_{(1\,2)} + e_{(1\,3)} + e_{(2\,3)} + e_{(1\,2\,3)} + e_{(1\,3\,2)})(e_1 - e_{(1\,4)}).$$

Similarly, one checks that

$$c_\lambda^* = (e_1 - e_{(1\,2)})(e_1 + e_{(1\,3)} + e_{(1\,4)} + e_{(3\,4)} + e_{(1\,3\,4)} + e_{(1\,4\,3)}).$$

We observe the following result.

Proposition 5.4. *The ideal $\mathbb{k}[\mathfrak{S}_n]c_\lambda^*$ is isomorphic to the Specht module S^λ.*

Proof. Observe that

$$c_{\lambda^\top}^* = \left(\sum_{\sigma \in P_\lambda} \operatorname{sgn}(\sigma) e_\sigma\right)\left(\sum_{\sigma \in Q_\lambda} e_\sigma\right).$$

If the coefficient of e_σ in c_λ is b, observe that the coefficient of e_σ in $c_{\lambda^\top}^*$ is $\operatorname{sgn}(\sigma)b$. Hence the module $\mathbb{k}[\mathfrak{S}_n]c_{\lambda^\top}^*$ yields a representation isomorphic to the tensor of $S^\lambda = \mathbb{k}[\mathfrak{S}_n]c_\lambda$ with the sign representation. By going from $\mathbb{k}[\mathfrak{S}_n]c_{\lambda^\top}^*$ to $\mathbb{k}[\mathfrak{S}_n]c_\lambda^*$, we once again tensor by the sign representation as we transpose the Young diagram. It follows that the representation due to the module $\mathbb{k}[\mathfrak{S}_n]c_\lambda^*$ is isomorphic to that given by the tensor of S^λ with two copies of the sign representation. Thus $\mathbb{k}[\mathfrak{S}_n]c_\lambda^* \cong S^\lambda$, as claimed. □

From now on, by Proposition 5.4 we may use the simpler formulation $S^\lambda \cong \mathbb{k}[\mathfrak{S}_n]c^*_\lambda$.

Definition 5.5. *Let $F_\lambda \in \mathbb{k}[x_1, x_2, \ldots, x_n]$ be the product of all binomials of the form $x_i - x_j$ for all pairs (i, j) in the same row of $t^{\lambda^\top}$, where $i < j$. Let $\mathbb{k}[\mathfrak{S}_n]F_\lambda$ be the right ideal generated by F_λ under the usual permutation action of $\mathbb{k}[\mathfrak{S}_n]$ on $\mathbb{k}[x_1, x_2, \ldots, x_n]$.*

Example 5.6. *If $\lambda = (3, 1)$, as before, then*

$$t^{(3,1)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

and

$$t^{(3,1)^\top} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}.$$

Then $F_{(3,1)} = x_1 - x_2$ while $F_{(3,1)^\top} = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$.

Proposition 5.7. *Define the monomial $m = \prod_{i=1}^n x_i^{a_i} \in \mathbb{k}[x_1, x_2, \ldots, x_n]$, where a_i is the number of cells strictly to the left of the cell with label i in $t^{\lambda^\top}$. Then $c^*_\lambda m = |P_\lambda| F_\lambda$.*

Proof. Suppose that i and j appear in the same row of $t^{\lambda^\top}$. Then $\langle (i\,j) \rangle$ is a subgroup of $\sigma \in P_{\lambda^\top}$. It thus follows that $e_1 - e_{(i\,j)}$ divides $\sum_{\sigma \in P_{\lambda^\top}} \operatorname{sgn}(\sigma) e_\sigma$ on the left. Hence, there exists some $c'' \in \mathbb{k}[\mathfrak{S}_n]$ such that $c^*_\lambda = (e_1 - e_{(i\,j)})c''$, so that $c^*_\lambda m = (e_1 - e_{(i\,j)})(c'' m)$. Specializing to $x_i = x_j$, it is immediate that the polynomial $c^*_\lambda m$ vanishes. Hence $x_i - x_j$ divides $c^*_\lambda m$ as long as i and j are in the same row of $t^{\lambda^\top}$, implying that F_λ divides $c^*_\lambda m$.

On the other hand, if $\lambda^\top = (b_1, b_2, \ldots, b_k)$ for $b_1 \geq b_2 \geq \cdots \geq b_k$, we note that

$$\deg F_\lambda = \sum_{i=1}^k \binom{b_i}{2} = \sum_{i=1}^k \sum_{j=0}^{b_j - 1} j = \sum_{i=1}^n a_i = \deg m.$$

As the action of $\mathbb{k}[\mathfrak{S}_n]$ only shuffles the terms, leaving degrees invariant, it follows that $c^*_\lambda m$ is multiple of F_λ by some $r \in \mathbb{k}$. To find r, observe that the value of a_i remains invariant over cells in the same column, so we have $\sigma m = m$ if and only if $\sigma \in Q_{\lambda^\top}$. Hence

$$c^*_\lambda m = |Q_{\lambda^\top}| \left(\sum_{\sigma \in P_{\lambda^\top}} \operatorname{sgn}(\sigma) e_\sigma \right) m.$$

Now by the same token the coefficient of m in $(\sum_{\sigma \in P_{\lambda^\top}} \operatorname{sgn}(\sigma) e_\sigma) m$ is exactly $|P_{\lambda^\top} \cap Q_{\lambda^\top}| = 1$, and hence the coefficient of m in $c^*_\lambda m$ is exactly $r = |Q_{\lambda^\top}| = |P_\lambda|$. $\square$

Proposition 5.8. *For any $z \in \mathbb{k}[\mathfrak{S}_n]$, the annihilation $zc^*_\lambda = 0$ holds if and only if $zF_\lambda = 0$.*

Proof. Let m be the same monomial from Proposition 5.7. If $zc^*_\lambda = 0$, then $|P_\lambda| zF_\lambda = zc^*_\lambda m = 0$, so that $zF_\lambda = 0$.

On the other hand, let $\psi \colon S^\lambda \to \mathbb{k}[\mathfrak{S}_n]F_\lambda$ satisfy $\psi(zc^*_\lambda) = zF_\lambda$. Note that this is well-defined, as if $z_1 c^*_\lambda = z_2 c^*_\lambda$ then $(z_1 - z_2)c^*_\lambda = 0$, implying by the above result that $(z_1 - z_2)F_\lambda = 0$, giving $z_1 F_\lambda = z_2 F_\lambda$. Now note that $\ker \psi$ is a proper subrepresentation of S^λ, from which irreducibility implies that $\ker \psi = \{0\}$. In other words, if $zF_\lambda = 0$, then $zc^*_\lambda = 0$, as desired. $\square$

A direct consequence of Proposition 5.8 is its implications of alternate formulations of Specht modules over fields of positive characteristic.

Corollary 5.9. *Over the field $\mathbb{F}_p$ of p elements, the representations S^λ and $\mathbb{F}_p[\mathfrak{S}_n]F_\lambda$ are isomorphic.*

Proof. By Proposition 5.8 on $\mathbb{C}$, we have that $zc_\lambda^* = 0$ if and only if $zF_\lambda = 0$ for all $z \in \mathbb{C}[\mathfrak{S}_n]$. Since restriction from $\mathbb{C}$ onto $\mathbb{Z}$ confers no additional algebraic relations, it follows that $zF_\lambda = 0$ if and only if $zc_\lambda^* = 0$ for all $z \in \mathbb{Z}[\mathfrak{S}_n]$. Hence, if we let $\psi' \colon \mathbb{Z}[\mathfrak{S}_n]c_\lambda^* \to \mathbb{Z}[\mathfrak{S}_n]F_\lambda$ be the pullback of ψ from $\mathbb{C}$ to $\mathbb{Z}$, we must have $\ker \psi' = \{0\}$, so ψ' is well-defined and injective. Since the preimage of any zF_λ always contains zc_λ^*, we note that ψ' is also surjective and hence a bijection. Finally, for any $z \in \mathbb{Z}[\mathfrak{S}_n]$ and $z_0 c_\lambda^* \in \mathbb{Z}[\mathfrak{S}_n]c_\lambda^*$, we have $\psi(z(z_0 c_\lambda^*)) = zz_0F_\lambda = z\psi(z_0c_\lambda^*)$, so ψ' intertwines with respect to the action of $\mathbb{Z}[\mathfrak{S}_n]$. We deduce that ψ' is an isomorphism.

It is well known that tensoring a module with $\mathbb{F}_p$ is equivalent to quotienting out every multiple of p in that module. Hence, over $\mathbb{F}_p$ the Specht module admits the characterization

$$S_\lambda = \mathbb{F}_p[\mathfrak{S}_n]c_\lambda^* \cong \mathbb{Z}[\mathfrak{S}_n]c_\lambda^* \otimes_\mathbb{Z} \mathbb{F}_p \cong \mathbb{Z}[\mathfrak{S}_n]F_\lambda \otimes_\mathbb{Z} \mathbb{F}_p \cong \mathbb{F}_p[\mathfrak{S}_n]F_\lambda$$

and we finish. □

As a result, rather than thinking about the Specht module S^λ in terms of $\mathbb{F}_p[\mathfrak{S}_n]$, we may instead do so in terms of polynomials in $\mathbb{F}_p[x_1, x_2, \ldots, x_n]$.

Remark 5.10. *One may naturally wonder whether Proposition 5.8 can be extended to fields of nonzero characteristic. We conjecture that the answer is in the positive.*

Observe that we exploit the characteristic of $\Bbbk$ solely to guarantee that $\ker\psi$ is trivial due to irreducibility in the latter half of our proof of Proposition 5.8. Hence only the direction that having $zF_\lambda = 0$ implies $zc_\lambda^* = 0$ still requires additional investigation.

5.2. **Exact sequence over characteristic two.** We bridge our work in Sections 4 and 5.1 by translating the exact sequences on the standard modules of $\mathsf{TL}_n(0)$ onto the Specht modules. However, instead of adapting base field $\mathbb{C}$, on which the Specht modules are irreducible, we shall instead do so over $\mathbb{F}_2$. By Corollary 5.9 on $p = 2$, we may work with the polynomial Specht modules. To introduce nilpotent elements, we shall also quotient by the set $\{x_1^2, x_2^2, \ldots, x_n^2\}$.

Fix some even $\ell \leq n$.

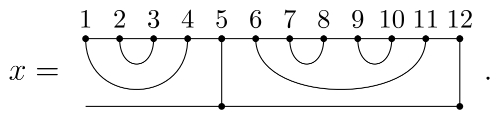
Definition 5.11. *Let $T^\lambda = \mathbb{F}_2[\mathfrak{S}_n]F_\lambda/\langle x_1^2, x_2^2, \ldots, x_n^2\rangle \subseteq \mathbb{F}_2[x_1, x_2, \ldots, x_n]/\langle x_1^2, x_2^2, \ldots, x_n^2\rangle$ be the Specht module on the algebra $\mathbb{F}_2[\mathfrak{S}]$ in which all squares are quotiented out.*

Definition 5.12. *Let $\lambda = (\frac{n+\ell}{2}, \frac{n-\ell}{2})$. Define the map $G_\ell^n \colon W_\ell^n \to T^\lambda$ such that, for some basis element $x \in W_\ell^n$, the image $G_\ell^n(x)$ consists of the product of all terms of the form $x_i + x_j$ for all (i, j) such that the ith and jth leftmost points on top are connected by a cup.*

Remark 5.13. *Since the representation of the (-1)-Specht modules on two-row partitions factors through the epimorphism $\theta \colon \mathcal{H}_n(q) \to \mathsf{TL}_n(\beta)$ from Proposition 2.8 to yield the standard modules of the Temperley-Lieb algebra, it follows automatically that the (-1)-Specht modules on two-row partitions form the structure of a long exact sequence. However, the structure behind such a sequence is perhaps unclear as the q-Specht modules are tractable only when $q = 1$, as then the Hecke algebra $\mathcal{H}_n(1)$ collapses into $\mathbb{C}[\mathfrak{S}_n]$. The motivation behind working over $\mathbb{F}_2$ thus follows* *from the fact that* $1 = -1$ *in a field of characteristic two, allowing us to write the* (-1)*-Specht modules as spans of Specht polynomials as established in Corollary 5.9.*

As discussed in Section 1, the Specht modules are irreducible over a field of characteristic zero, equivalent to having q *not being a root of unity. This is due to the philosophy that working in a field of prime characteristic* p *is tantamount to working with* $q = e^{2\pi i a/p}$ *for some* $p \nmid a$ [30]. *Our situation falls in the case where* $p = 2$.

Example 5.14. *Consider the element* $x \in W_2^{12}$ *given by*



Since nodes 1 *and* 4, 2 *and* 3, 6 *and* 11, 7 *and* 8, *and* 9 *and* 10 *are connected by cups, we have*

$$G_2^{12}(x) = (x_1 + x_4)(x_2 + x_3)(x_6 + x_{11})(x_7 + x_8)(x_9 + x_{10}).$$

Proposition 5.15. *The map* G_ℓ^n *is a bijection on vector spaces.*

Proof. One checks by the hook length formula that

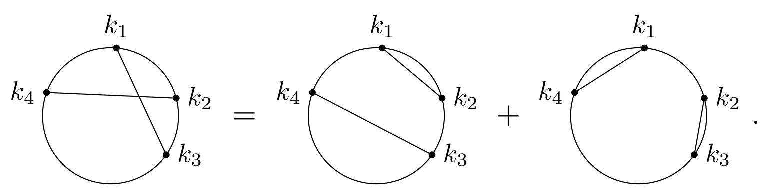
$$\dim S^\lambda = \binom{n}{\frac{n-\ell}{2}} - \binom{n}{\frac{n-\ell}{2} - 1} = \dim W_\ell^n.$$

We defer the calculations in full to Section 6. Thus it suffices to show that G_ℓ^n is a surjection.

Since λ consists of at most two columns, any basis element $w \in T^\lambda$ is a product of binomials. We will represent such a polynomial by writing the numbers from 1 to n such that they are evenly spaced around a circle. Draw a chord from i to j if and only if $x_i + x_j$ divides w. Observe for any k_1, k_2, k_3, and k_4 that

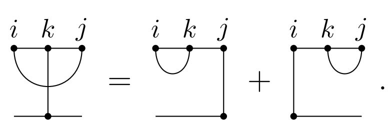
$$(x_{k_1} + x_{k_3})(x_{k_2} + x_{k_4}) = (x_{k_1} + x_{k_2})(x_{k_3} + x_{k_4}) + (x_{k_1} + x_{k_4})(x_{k_2} + x_{k_3})$$

in $\mathbb{F}_2[x_1, x_2, \ldots, x_n]/\langle x_1^2, x_2^2, \ldots, x_n^2 \rangle$, implying that any intersection of chords can be resolved by



The total length of the chords drawn on each component must strictly decrease every time we use the above resolution. Hence, using a finite number of resolutions, we may write our original starting combination of chords, corresponding to w, as a sum of chord diagrams such that no two chords intersect. Each chord diagram can then be unraveled to yield a diagram with n points on top and a number of nonintersecting cups connecting pairs of such points. We draw in the throughlines, which pass through all upper marked points that are not connected by cups.

We still need to consider the possibility that a throughline may intersect an arc. If the cup corresponding to $x_i + x_j$ intersects the throughline corresponding to x_k, then we have $x_i + x_j = (x_i + x_k) + (x_j + x_k)$, giving us the resolution



Note that every use of the above resolution decreases the number of crossings by 1. No new crossings are introduced as the throughlines are all vertical. We may thus write w as a sum of polynomials corresponding to such diagrams of cups and throughlines, with no intersections at all. However, such diagrams are exactly the basis elements of W_ℓ^n, and can be mapped to a polynomial in T^λ via G_ℓ^n. In other words, w lies in the image of G_ℓ^n. ☐

Definition 5.16. *Let* $\psi_\ell^n \colon T^{(\frac{n+\ell}{2}+1, \frac{n-\ell}{2}-1)} \to T^{(\frac{n+\ell}{2}, \frac{n-\ell}{2})}$ *act by multiplication by* $\sum_{i=1}^n x_n$.

Proposition 5.17. *The map* ϕ_ℓ^n *is the lift of* ψ_ℓ^n*, so that* $\psi_\ell^n \circ G_{\ell+2}^n = G_\ell^n \circ \phi_\ell^n$*. In particular, the following diagram commutes.*

$$\begin{array}{ccc} W_{\ell+2}^n & \xrightarrow{\phi_\ell^n} & W_\ell^n \\ {\scriptstyle G_{\ell+2}^n}\big\downarrow & & \big\downarrow{\scriptstyle G_\ell^n} \\ T^{(\frac{n+\ell}{2}+1, \frac{n-\ell}{2}-1)} & \xrightarrow{\psi_\ell^n} & T^{(\frac{n+\ell}{2}, \frac{n-\ell}{2})} \end{array}$$

Proof. Consider a basis element $x \in W_{\ell+2}^n$. Number the points on the top row of the diagrammatic representation of x with the integers from 1 to n, going from left to right. For each $j \le \ell+2$, suppose that the jth leftmost throughline occurs at the point numbered with k_j. Then $G_\ell^n \circ \phi_\ell^n$ takes an alternating sum over connecting the $(2i-1)$th and $2i$th throughlines with a cup, which multiplies the Specht polynomial $G_{\ell+2}^n(x)$ with the binomial $x_{k_{2i-1}} + x_{k_{2i}}$. In characteristic 2 the alternating sum becomes a normal sum, and it follows that

$$G_\ell^n(\phi_\ell^n(x)) = \sum_{i=1}^{\ell/2+1} (x_{k_{2i-1}} + x_{k_{2i}}) G_{\ell+2}^n(x) = \sum_{i=1}^{\ell+2} x_{k_i} G_{\ell+2}^n(x).$$

Note that $G_{\ell+2}^n(x) = \prod_{i=1}^{(n-\ell)/2-1} b_i$ where the binomials b_i have $\sum_{i=1}^{(n-\ell)/2-1} b_i + \sum_{i=1}^{\ell+2} x_{k_i} = \sum_{i=1}^n x_n$. Since in $\mathbb{F}_2[x_1, x_2, \dots, x_n]/\langle x_1^2, x_2^2, \dots, x_n^2\rangle$ we have $b_i^2 = 0$ for all i, it follows that

$$\left(\sum_{i=1}^n x_n - \sum_{i=1}^{\ell+2} x_{k_i}\right) G_{\ell+2}^n(x) = \sum_{i=1}^{(n-\ell)/2-1} b_i G_{\ell+2}^n(x) = 0.$$

Combining the above equations implies that

$$\psi_\ell^n(G_{\ell+2}^n(x)) = \sum_{i=1}^n x_n G_{\ell+2}^n(x) = \sum_{i=1}^{\ell+2} x_{k_i} G_{\ell+2}^n(x) = G_\ell^n(\phi_\ell^n(x)),$$

as claimed. ☐

We can now complete the proof of Theorem 1.3.

Corollary 5.18 (Theorem 1.3)**.** *The* ϕ_ℓ^n *constitute a long exact sequence of homomorphisms between adjacent standard modules* T^λ *on* $\mathbb{F}_2[x_1, x_2, \dots, x_n]/\langle x_1^2, x_2^2, \dots, x_n^2\rangle$ *for two-row partitions, given by*

$$0 \xrightarrow{\psi_n^n} T^{(n)} \xrightarrow{\psi_{n-2}^n} T^{(n-1,1)} \xrightarrow{\psi_{n-4}^n} \cdots \xrightarrow{\psi_2^n} T^{(n/2+1, n/2-1)} \xrightarrow{\psi_0^n} T^{(n/2,n/2)} \longrightarrow 0.$$

Proof. Recall the exactness of the ϕ_ℓ^n due to Theorem 4.6. Then the desired result is immediate by Propositions 5.15 and 5.17. ☐

6. Proof of Proposition 1.4 and applications

Fix a positive integer n. We start by giving a quick refresher of the connections between the Hecke algebra and braid group that we have mentioned previously in Section 2.

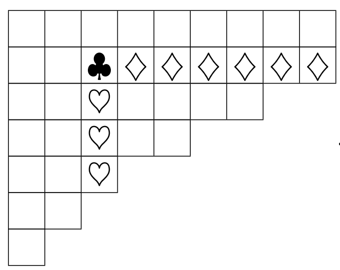
Definition 6.1. *When q is not a root of unity, the irreducible modules of the Hecke algebra $\mathcal{H}_n(q)$ are the q-Specht modules, denoted by S_q^λ, and are indexed by partitions $\lambda \vdash n$.*

Definition 6.2. *Let $\pi\colon B_n \to \mathcal{H}_n(q)$ be the homomorphism such that $\pi(\sigma_i) = g_i$ for all i, and let $\chi_\lambda\colon \mathcal{H}_n(q) \to \mathbb{C}$ be the character of the q-Specht module S_q^λ as a representation of $\mathcal{H}_n(q)$. Also, let $e\colon B_n \to \mathbb{Z}$ be the homomorphism such that $e(\sigma_i) = 1$ for all i. For any $\alpha \in B_n$, we call $e(\alpha)$ its exponent sum.*

Fittingly, under specialization to $q = 1$, the q-Specht modules collapse into the usual Specht modules defined in Section 5.1 as $\mathcal{H}_n(q)$ collapses into the group algebra $\mathbb{C}[\mathfrak{S}_n]$.

Definition 6.3. *Let $\lambda \vdash n$ be a partition. For a box $b \in \lambda$ in its Young diagram, let the hook length of b, denoted $h(b)$, be the number of boxes directly below or to the right to b, including b itself. Also, let $r(b)$ be the number of rows below the topmost row that b resides in, and define $c(b)$ analogously but in comparison to the leftmost column.*

Example 6.4. *Consider the partition*

$(9, 9, 7, 5, 3, 2, 1) =$  .

The box b marked with a club has 3 boxes below it, denoted using hearts, and 6 boxes to its right, denoted using diamonds. Counting the boxes, the hook length of b is then $h(b) = 3 + 6 + 1 = 10$. We can also check that $r(b) = 1$ and $c(b) = 2$.

Proposition 6.5 ([20], Definition 3.5)**.** *The dimension of the q-Specht module is given by*

$$\dim S_q^\lambda = \frac{n!}{\prod_{b\in\lambda} h(\lambda)}.$$

Corollary 6.6. *Let $\lambda = (n-k, k)$ be a two-row partition for some $k \leq \frac{n}{2}$. Then*

$$\dim S_q^\lambda = \binom{n}{k} - \binom{n}{k-1}.$$

Proof. Define $\ell = n - 2k$. We note that λ is given by a $1 \times \ell$ rectangular grid attached to the right of a $2 \times k$ rectangular grid. In the $1 \times \ell$ rectangular grid, the hook length ranges from 1 to ℓ. Similarly, in the rightmost column of the $2 \times k$ grid, the hook length ranges from 1 to k, while in the leftmost column of the same grid the hook length ranges from $\ell + 2$ to $k + \ell + 1$. We therefore have the multiset equality

$$\{h(b) \mid b \in \lambda\} = (\{i \mid 1 \leq i \leq k + \ell + 1\} \setminus \{\ell + 1\}) \cup \{i \mid 1 \leq i \leq k\}.$$

Hence Proposition 6.5 yields that

$$\dim S_q^\lambda = \frac{n!(\ell+1)}{(k+\ell+1)!k!} = \binom{n}{k}\left(1 - \frac{k}{n-k+1}\right) = \binom{n}{k} - \binom{n}{k-1},$$

as claimed. □

This confirms the dimension equivalence claimed at the start of the proof of Proposition 5.15. Note that setting $\ell = n - 2k$ is apt, as W_ℓ^n has the same dimension as $S^{(\frac{n+\ell}{2}, \frac{n-\ell}{2})} = S^{(n-k,k)}$.

Now consider the two-variable link polynomial $X_L(q, \Lambda)$ in the context of braid closures. To avoid confusion with the partitions $\lambda \vdash n$, we shall use the capital letter Λ to denote the second parameter of $X_L(q, \Lambda)$, instead of the more standard λ.

Definition 6.7 ([20], Definition 6.1)**.** *For a braid $\alpha \in B_n$, the HOMFLY polynomial under Jones normalization*

$$X_{\hat{\alpha}}(q, \Lambda) = \left(-\frac{1-\Lambda q}{\sqrt{\Lambda}(1-q)}\right)^{n-1} (\sqrt{\Lambda})^{e(\alpha)} \operatorname{tr}(\pi(\alpha))$$

is an invariant of the oriented braid closure link $\hat{\alpha}$. Here $\operatorname{tr}(\pi(\alpha))$, also known as the Ocneanu trace, is the character of the regular representation of $\mathcal{H}_n(q)$ evaluated at α. Moreover, the Jones polynomial is obtained by the specialization $V_{\hat{\alpha}}(t) = X_{\hat{\alpha}}(t, t)$.

We can also write $\operatorname{tr}(\pi(\alpha))$ in terms of characters of the q-Specht modules.

Theorem 6.8 ([15], Theorem 1.1)**.** *For $\alpha \in B_n$, the Ocneanu trace $\pi(\alpha)$ is given by the sum*

$$\operatorname{tr}(\pi(\alpha)) = \sum_{\lambda \vdash n} \Omega_\lambda \chi_\lambda(\alpha),$$

where setting $z = -\frac{1-q}{1-\Lambda q}$ and $w = 1 - q + z$, the weight Ω_λ satisfies

$$\Omega_\lambda = \prod_{b \in \lambda} \frac{q^{r(b)} w - q^{c(b)} z}{1 - q^{h(b)}}.$$

Here, the product runs through all n boxes $b \in \lambda$.

Let Ω_λ^* be the rational function in t given by specializing the weight Ω_λ on $q = \Lambda = t$. We now prove the following results that will be key to recovering the Jones polynomial of n-braids.

Proposition 6.9. *The weight of S_q^λ under the specialization $q = \Lambda = t$ is given by*

$$\Omega_\lambda^* = \frac{1}{(1+t)^n} \prod_{b \in \lambda} \frac{t^{c(b)} - t^{r(b)+2}}{1 - t^{h(b)}}.$$

Proof. Comparing formulas for Ω_λ and Ω_λ^*, note that it suffices to show that

$$q^{r(b)} w - q^{c(b)} z = \frac{t^{c(b)} - t^{r(b)+2}}{1+t}$$

if $q = \Lambda = t$. This follows as we find that $z = -\frac{1-q}{1-\Lambda q} = -\frac{1-t}{1-t^2} = -\frac{1}{1+t}$, giving

$$q^i w - q^j z = q^i(1 - q + z) - q^j z = q^i(1-q) + z(q^i - q^j) = t^i(1-t) - \frac{t^i - t^j}{1+t} = \frac{t^j - t^{i+2}}{1+t}.$$

We finish by substituting in $(i, j) = (r(b), c(b))$. □

Proposition 6.10. *For $k < \frac{n}{2}$, let $\lambda = (n-k,k)^\top$ be a partition with at most two columns. Then*

$$\Omega^*_\lambda = \frac{1}{(1+t)^n}\sum_{i=k}^{n-k} t^i.$$

Proof. Again define $\ell = n-2k$. Looking at the two possible columns of λ, we observe the multiset equivalence

$$\{(r(b),c(b)) \mid b\in\lambda\} = \{(i,0)\mid 0\le i\le k+\ell-1\}\cup\{(i,1)\mid 0\le i\le k-1\}.$$

Since the set of hook lengths is invariant when taking transposes, by Corollary 6.6 we have that

$$\{h(b)\mid b\in\lambda\} = (\{i\mid 1\le i\le k+\ell+1\}\setminus\{\ell+1\})\cup\{i\mid 1\le i\le k\}.$$

By Proposition 6.9, it follows that

$$\begin{aligned}
\Omega^*_\lambda &= \frac{1}{(1+t)^n}\prod_{b\in\lambda}\frac{t^{c(b)}-t^{r(b)+2}}{1-t^{h(b)}}\\
&= \frac{1}{(1+t)^n}\prod_{i=0}^{k+\ell-1}(1-t^{i+2})\prod_{i=1}^{k-1}(t-t^{i+2})\left(\frac{1}{1-t^{\ell+1}}\prod_{i=1}^{k+\ell+1}(1-t^i)\prod_{i=1}^{k}(1-t^i)\right)^{-1}\\
&= \frac{t^k(1-t^{\ell+1})}{(1+t)^n}\prod_{i=2}^{k+\ell+1}(1-t^i)\prod_{i=1}^{k}(1-t^i)\left(\prod_{i=1}^{k+\ell+1}(1-t^i)\prod_{i=1}^{k}(1-t^i)\right)^{-1}\\
&= \frac{t^k(1-t^{\ell+1})}{(1+t)^n(1-t)}\\
&= \frac{1}{(1+t)^n}\sum_{i=k}^{n-k}t^i
\end{aligned}$$

as claimed. □

We are now ready to prove our main result.

Proposition 6.11 (Proposition 1.4)**.** *The braid closure of any $\alpha\in B_n$ has Jones polynomial*

$$V_{\hat\alpha}(t) = \frac{(-1)^{n-1}(\sqrt{t})^{e(\alpha)-n+1}}{1+t}\sum_{k=0}^{\lfloor n/2\rfloor}\left(\sum_{i=k}^{n-k}t^i\right)\chi_{(n-k,k)^\top}(\alpha).$$

Proof. If a partition has more than two columns, then there exists some box $b\in\lambda$ satisfying $r(b)=0$ and $c(b)=2$. Thus Ω^*_λ contains a factor of $w-q^2z = \frac{t^2-t^2}{1+t}=0$ by Proposition 6.9 and is thus equal to 0.

Hence only partitions with at most two columns, that is, of the form $(n-k,k)^\top$ for $k<\frac{n}{2}$, contribute to the trace. Applying Proposition 6.10, we find that

$$\operatorname{tr}(\pi(\alpha)) = \frac{1}{(1+t)^n}\sum_{k=0}^{\lfloor n/2\rfloor}\left(\sum_{i=k}^{n-k}t^i\right)\chi_{(n-k,k)^\top}(\alpha).$$

Specializing the HOMFLY polynomial to obtain the Jones polynomial, by Theorem 6.7 we get

$$\begin{aligned}V_{\hat{\alpha}}(t) &= X_{\hat{\alpha}}(t,t)\\ &= \left(-\frac{1-t^2}{\sqrt{t}(1-t)}\right)^{n-1}(\sqrt{t})^{e(\alpha)}\operatorname{tr}(\pi(\alpha))\\ &= ((-1)^{n-1}(\sqrt{t})^{e(\alpha)-n+1}(1+t)^{n-1})\left(\frac{1}{(1+t)^n}\sum_{k=0}^{\lfloor n/2\rfloor}\left(\sum_{i=k}^{n-k}t^i\right)\chi_{(n-k,k)^\top}(\alpha)\right)\\ &= \frac{(-1)^{n-1}(\sqrt{t})^{e(\alpha)-n+1}}{1+t}\sum_{k=0}^{\lfloor n/2\rfloor}\left(\sum_{i=k}^{n-k}t^i\right)\chi_{(n-k,k)^\top}(\alpha),\end{aligned}$$

as desired. □

Example 6.12. *We apply Lemma 6.11 to the case where $n = 3$. Here, we observe that*

$$\begin{aligned}V_{\hat{\alpha}}(t) &= \frac{(-1)^{n-1}(\sqrt{t})^{e(\alpha)-n+1}}{1+t}\sum_{k=0}^{\lfloor n/2\rfloor}\left(\sum_{i=k}^{n-k}t^i\right)\chi_{(n-k,k)^\top}(\alpha)\\ &= \frac{(\sqrt{t})^{e(\alpha)}((1+t+t^2+t^3)\chi_{(1,1,1)}(\alpha)+(t+t^2)\chi_{(2,1)}(\alpha))}{t(1+t)}\\ &= (\sqrt{t})^{e(\alpha)}\left(\left(t+\frac{1}{t}\right)\chi_{(1,1,1)}(\alpha)+\chi_{(2,1)}(\alpha)\right).\end{aligned}$$

The (-1)-Specht module $S_{-1}^{(1,1,1)}$ yields the sign representation, so $\chi_{(1,1,1)}(\alpha) = (-1)^{e(\alpha)}$ and

$$V_{\hat{\alpha}}(t) = (\sqrt{t})^{e(\alpha)}\left(\left(t+\frac{1}{t}\right)(-1)^{e(\alpha)}+\chi_{(2,1)}(\alpha)\right).$$

This exactly verifies the formula for the Jones polynomial of a braid closure $\hat{\alpha}$ for $\alpha \in B_3$ as claimed by Birman [5].

Remark 6.13. *If we specialize to $t = -1$, then Theorem 6.11 must express $V_{\hat{\alpha}}(t)$ as an indeterminate form. In other words, we expect at $t = -1$ that*

$$1+t = \sum_{k=0}^{\lfloor n/2\rfloor}\left(\sum_{i=k}^{n-k}t^i\right)\chi_{(n-k,k)^\top}(\alpha) = 0.$$

The above is guaranteed for odd n as then

$$\sum_{i=k}^{n-k}(-1)^i = (-1)^k\sum_{i=0}^{n-2k}(-1)^i = 0.$$

For even n, however, it must instead be the case that

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\chi_{(n-k,k)^\top}(\alpha) = 0.$$

After tensoring with the sign representation to transpose the two-column partitions $(n-k,k)^\top$ to their corresponding two-row partitions $(n-k,k)$, it is not surprising that there exists a long exact sequence on the Specht modules $T^{(n-k,k)}$ over $\mathbb{F}_2[x_1, x_2, \ldots, x_n]/\langle x_1^2, x_2^2, \ldots, x_n^2\rangle$ as we have shown in Theorem 5.18.

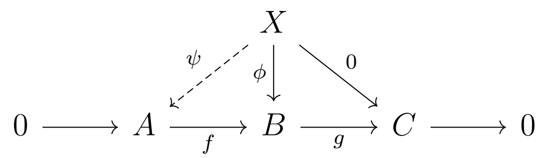
In general, the q-Specht modules are irreducible whenever q is not a root of unity. This corresponds directly to the case where the standard modules W_ℓ^n of the Temperley-Lieb algebra are irreducible whenever we have $\beta = q^{1/2} + q^{-1/2}$ for q not a root of unity.

Appendix A. Proof of Lemma 3.15 with diagram chasing

In this section, we develop some useful representation-theoretic background and compute several homomorphism spaces between the modules of $\mathsf{TL}_n(0)$ using rather elementary methods.

Throughout this section, we fix some field $\Bbbk$ and designate A as a general $\Bbbk$-algebra.

Proposition A.1. *Let A, B, C, and X be A-modules such that A, B, and C form a short exact sequence and let $\phi\colon X \to B$ be a homomorphism as in the following diagram.*

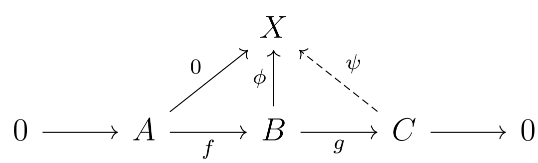


Suppose that $g \circ \phi = 0$. Then ϕ factors through A as $f \circ \psi$ for some homomorphism $\psi\colon X \to A$.

Proof. Let $x \in X$, so that $g(\phi(x)) = 0$. By exactness $\phi(x) \in \ker g = \operatorname{im} f$. Since f is injective there exists some unique $y \in A$ such that $\phi(x) = f(y)$. Thus we may make the assignment $\psi(x) = y$ as a map of vector spaces. We check that $\phi = f \circ \psi$, and it follows that ψ is a valid homomorphism because it certainly intertwines. □

In fact, the dual of Proposition A.1 holds true as well.

Proposition A.2. *Let A, B, C, and X be A-modules such that A, B, and C form a short exact sequence and let $\phi\colon B \to X$ be a homomorphism as in the following diagram.*



Suppose that $\phi \circ f = 0$. Then ϕ factors through C as $\psi \circ g$ for some homomorphism $\psi\colon C \to X$.

Proof. By surjectivity, for any $c \in C$ there exists $b_0 \in B$ such that $g(b_0) = c$. Let $\psi(c) = \phi(b_0)$, so it is immediate that $\phi = \psi \circ g$. Clearly ψ intertwines and thus is a homomorphism.

For all $b \in \ker g$, we have by exactness that $b \in \operatorname{im} f$ and thus $b = f(a)$ for some $a \in A$, giving $\phi(b) = \phi(f(a)) = 0$. Hence if $g(b_1) = g(b_2)$ then $b_1 - b_2 \in \ker g \subseteq \ker \phi$, so $\phi(b_1) = \phi(b_2)$. It follows that ψ is well-defined. □

Now we formally define the notion of composition series.

Definition A.3. *Let V be an A-module of finite dimension. Suppose that there exist submodules $V_0, V_1, V_2, \ldots, V_k$ with $V_0 = V$ and $V_k = \{0\}$ such that there exists a filtration*

$$\{0\} = V_k \subseteq V_{k-1} \subseteq \cdots \subseteq V_1 \subseteq V_0 = V$$

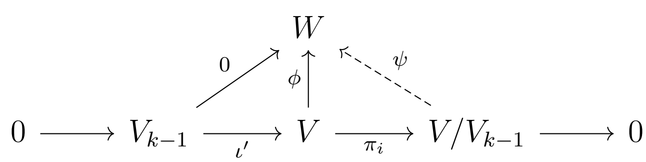
for which V_i/V_{i+1} is irreducible for all $i \leq k-1$. Then the sequence $(V_0, V_1, \ldots, V_k)$ is a composition series of V, with irreducibles of the form V_i/V_{i+1} called its subquotients. The following facts about composition series are well-known [13].

- *The Jordan-Hölder Theorem states that if* $(V_0, V_1, \ldots, V_k)$ *and* $(V'_0, V'_1, \ldots, V'_{k'})$ *are two composition series of* V*, then we have the multiset equality*
$$\{V_0/V_1, V_1/V_2, \ldots, V_{k-1}/V_k\} = \{V'_0/V'_1, V'_1/V'_2, \ldots, V'_{k'-1}/V'_k\}.$$
- *If* $U \subseteq V$ *is a submodule of* V*, then there exists a composition series* $(V_0, V_1, \ldots, V_k)$ *of* V *containing* U*, that is,* $U \in \{V_0, V_1, \ldots, V_k\}$*.*

Proposition A.4. *Let* V *and* W *be* A*-modules with* V *having composition series* $(V_0, V_1, \ldots, V_k)$ *and* W *having composition series* $(W_0, W_1, \ldots, W_m)$*. Suppose for some* i *and* j *that*
$$\{V_i/V_{i+1}, V_{i+1}/V_{i+2}, \ldots, V_{k-1}/V_k\} \cap \{W_0/W_1, W_1/W_2, \ldots, W_{m-1}/W_m\} = \emptyset$$
and
$$\{W_0/W_1, W_1/W_2, \ldots, W_{j-1}/W_j\} \cap \{V_0/V_1, V_1/V_2, \ldots, V_{k-1}/V_k\} = \emptyset.$$
Then every homomorphism $\phi\colon V \to W$ *factors through as* $\phi = \iota_j \circ \psi \circ \pi_i$ *where* $\iota_j\colon W_j \to W$ *is the simple inclusion and* $\pi_i\colon V \to V/V_i$ *is the simple projection.*

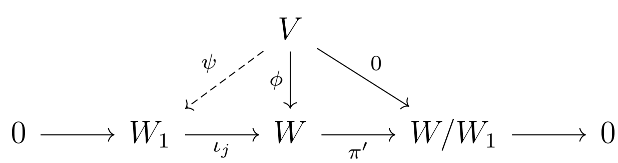
Proof. We show the desired result for $i = k-1$ and when $j = 1$, as we may iterate inductively for larger values of i and j.

Let $i = k-1$. Suppose that $V_{k-1} \notin \{W_0/W_1, W_1/W_2, \ldots, W_{m-1}/W_m\}$. Then there is an inclusion $\iota'\colon V_{k-1} \to V$, with which $\phi' = \phi \circ \iota'$ is a homomorphism from V_{k-1} to W. Since V_{k-1} is irreducible, by Schur's Lemma it follows that $\ker \phi'$ is either $\{0\}$ or V_{k-1}. The former case implies that V_{k-1} is a subrepresentation of W. Hence there is a composition series of W containing V_{k-1}, from which the Jordan-Hölder yields $V_{k-1} \in \{W_0/W_1, W_1/W_2, \ldots, W_{m-1}/W_m\}$, a contradiction. It follows that $\ker \phi' = V_{k-1}$ and thus $\phi' = 0$, and we have the following diagram.



In particular, Proposition A.2 implies that ϕ factors through as $\psi \circ \pi_i$ as claimed.

Now suppose instead that $j = 1$. We have $W/W_1 \notin \{V_0/V_1, V_1/V_2, \ldots, V_{k-1}/V_k\}$. There is a projection $\pi'\colon W \to W/W_1$, from which $\phi'' = \pi' \circ \phi$ is a homomorphism from V to W/W_1. Since W/W_1 is irreducible, we note that $\operatorname{im} \phi''$ is either $\{0\}$ or W/W_1. In the latter case, we have an epimorphism from V onto W/W_1. However, applying our work from the above case of $i = k-1$, this cannot happen, since $W/W_1 \notin \{V_0/V_1, V_1/V_2, \ldots, V_{k-1}/V_k\}$ implies that the epimorphism must factor through all projections. Thus $\phi'' = 0$, yielding the following diagram.



Using Proposition A.1, we conclude that ϕ factors through as $\iota_j \circ \psi$. ☐

Note that Proposition A.4 formalizes the subquotient multiplicity argument that we used in the latter half of the proof of Theorem 3.9.

We now demonstrate some applications of the above results by computing some homomorphism spaces between the standard and projective modules of $\mathsf{TL}_n(0)$.

Proposition A.5. *The homomorphism space* $\mathrm{Hom}(W_\ell^n, W_{\ell+2}^n)$ *is trivial.*

Proof. Let $\phi\colon W_\ell^n \to W_{\ell+2}^n$ be a homomorphism. Corollary 4.8 yields the short exact sequences

$$0 \longrightarrow L_{\ell+2}^n \longrightarrow W_\ell^n \longrightarrow L_\ell^n \longrightarrow 0$$

and

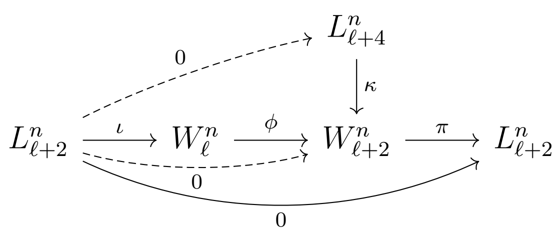
$$0 \longrightarrow L_{\ell+4}^n \longrightarrow W_{\ell+2}^n \longrightarrow L_{\ell+2}^n \longrightarrow 0.$$

Hence there is an inclusion $\iota\colon L_{\ell+2}^n \to W_\ell^n$ and a projection $\pi\colon W_{\ell+2}^n \to L_{\ell+2}^n$. Thus $\pi \circ \phi \circ \iota$ is an endomorphism of $L_{\ell+2}^n$ and must therefore be scalar multiplication by Schur's Lemma. If this composition is not nonzero, then $\phi \circ \iota$ is a section map, implying that $W_{\ell+2}^n$ splits into direct summands, which is a contradiction as it is well-known [28] that $W_{\ell+2}$ is indecomposable. Hence $\pi \circ \phi \circ \iota$ must be the zero map.

Let $\kappa\colon L_{\ell+4}^n \to W_{\ell+2}^n$ be the inclusion map from the short exact sequence

$$0 \longrightarrow L_{\ell+4}^n \longrightarrow W_{\ell+2}^n \longrightarrow L_{\ell+2}^n \longrightarrow 0.$$

We thus have the following diagram.

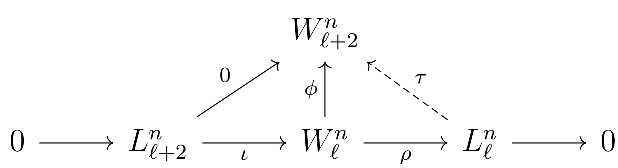


Specifically, applying Proposition A.1 implies that $\phi \circ \iota$ factors through $L_{\ell+4}^n$ as above, but by Schur's Lemma $\mathrm{Hom}(L_{\ell+2}^n, L_{\ell+4}^n) = 0$. Thus $\phi \circ \iota = \kappa \circ 0 = 0$.

Let $\rho\colon W_\ell^n \to L_\ell^n$ be the projection map from the short exact sequence

$$0 \longrightarrow L_{\ell+2}^n \longrightarrow W_\ell^n \longrightarrow L_\ell^n \longrightarrow 0.$$

Now our diagram is the following.



Proposition A.2 implies that ϕ factors through L_ℓ^n so that $\phi = \tau \circ \rho$. Here $\tau \in \mathrm{Hom}(L_\ell^n, W_{\ell+2}^n)$. Since Corollary 4.8 implies that $W_{\ell+2}^n$ has composition series $0 \subsetneq L_{\ell+4}^n \subsetneq W_{\ell+2}^n$, we find that $W_{\ell+2}^n$ has subquotient multiset $\{L_{\ell+2}^n, L_{\ell+4}^n\}$. Observe that $L_\ell^n \notin \{L_{\ell+2}^n, L_{\ell+4}^n\}$, from which Proposition A.4 implies that τ factors through the projection and inclusion maps and must therefore be zero. We conclude that $\phi = 0 \circ \rho = 0$. $\square$

Proposition A.6. *If* $|\ell - m| \geq 6$ *or* $m = \ell + 4$*, then* $\mathrm{Hom}(P_\ell, P_m)$ *is trivial.*

Proof. Recall by Proposition 3.5, we have the exact sequence

$$0 \longrightarrow W_{\ell-2}^n \longrightarrow P_\ell^n \longrightarrow W_\ell^n \longrightarrow 0.$$

Hence there is a sequence of inclusions

$$0 \subseteq W_{\ell-2}^n \subseteq P_\ell^n$$

where $P_\ell^n/W_{\ell-2}^n \cong W_\ell^n$. Similarly, by Corollary 4.8, we have the inclusion

$$0 \subseteq L_{\ell+2}^n \subseteq W_\ell^n$$

where $W_\ell^n/L_{\ell+2}^n \cong L_\ell^n$. Thus by Lemma 3.10 there exists a space V such that the series of inclusions

$$0 \subseteq L_\ell^n \subseteq W_{\ell-2}^n \subseteq V \subseteq P_\ell^n$$

forms a composition series such that the subquotients are L_ℓ^n, $W_{\ell-2}^n/L_\ell^n \cong L_{\ell-2}^n$, $V/W_{\ell-2}^n \cong L_{\ell+2}^n$, and $P_\ell^n/V \cong L_\ell^n$. The multiset of irreducible subquotients on P_ℓ^n is thus $\{L_\ell^n, L_{\ell-2}^n, L_{\ell+2}^n, L_\ell^n\}$.

If $|\ell - m| \geq 6$, the multisets of irreducible subquotients of P_ℓ^n and P_m^n are disjoint. By Proposition A.4, we can then factor through all elements of the composition series of the projectives, implying that $\mathrm{Hom}(P_\ell^n, P_m^n)$ is trivial.

In general, we have the composition series

$$0 = V_4 \subseteq V_3 \subseteq V_2 \subseteq V_1 \subseteq V_0 = P_\ell^n$$

with $V_2 = W_{\ell-2}^n$ and $V_3 = L_\ell^n$ such that

$$(V_0/V_1, V_1/V_2, V_2/V_3, V_3/V_4) = (L_\ell^n, L_{\ell+2}^n, L_{\ell-2}^n, L_\ell^n).$$

For $m = \ell + 4$, this gives

$$0 = W_4 \subseteq W_3 \subseteq W_2 \subseteq W_1 \subseteq W_0 = P_{\ell+4}^n$$

where $W_2 = W_{\ell+2}^n$ such that

$$(W_0/W_1, W_1/W_2, W_2/W_3, W_3/W_4) = (L_{\ell+4}^n, L_{\ell+6}^n, L_{\ell+2}^n, L_{\ell+4}^n).$$

Then

$$\{V_2/V_3, V_3/V_4\} \cap \{W_0/W_1, W_1/W_2, W_2/W_3, W_3/W_4\} = \emptyset$$

and

$$\{W_0/W_1, W_1/W_2\} \cap \{V_0/V_1, V_1/V_2, V_2/V_3, V_3/V_4\} = \emptyset,$$

so Proposition A.4 implies that any map from P_ℓ^n to $P_{\ell+4}^n$ factors through as $\phi = \iota_j \circ \psi \circ \pi_i$ in which $i = j = 2$. Then ψ is a map from $V_0/V_2 = P_\ell^n/W_{\ell-2}^n = W_\ell^n$ to $W_2 = W_{\ell+2}^n$. However, by Proposition A.5 we are forced to have $\psi \in \mathrm{Hom}(W_\ell^n, W_{\ell+2}^n)$ be the zero map, implying that ϕ must be zero as well. This finishes the proof. ☐

Observe that Proposition A.6 alone establishes a significant portion of the homomorphism space computations as done in Theorem 3.9. In the proof of the latter we use Frobenius reciprocity, which is fairly powerful as $\mathsf{TL}_{n-1}(0)$ is semisimple by Corollary 3.7. Here we instead found elementary solutions using primarily diagram chasing and the notion of composition series.

We close this section with the proof of Lemma 3.15. Recall from earlier that we have set $\omega_\ell^n \colon P_\ell^n \to P_{\ell+2}^n$ and $\gamma_\ell^n \colon P_{\ell+2}^n \to P_\ell^n$ to be the unique maps, up to a constant, between adjacent projectives, in which they attain the structure

$$P_2 \underset{\gamma_2^n}{\overset{\omega_2^n}{\rightleftarrows}} P_4 \underset{\gamma_4^n}{\overset{\omega_4^n}{\rightleftarrows}} P_6 \underset{\gamma_6^n}{\overset{\omega_6^n}{\rightleftarrows}} \cdots \underset{\gamma_{n-2}^n}{\overset{\omega_{n-2}^n}{\rightleftarrows}} P_n.$$

Lemma A.7 (Lemma 3.15)**.** *For every $\ell < n$, the compositions $\gamma_\ell^n \circ \omega_\ell^n$ and $\omega_\ell^n \circ \gamma_\ell^n$ are nonzero and nonidentity.*

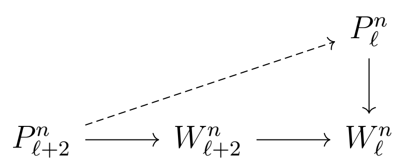
Proof. Let us first consider the composition $\gamma_\ell^n \circ \omega_\ell^n$. Again recall by Proposition 3.5, that we have the exact sequence

$$0 \longrightarrow W_{\ell-2}^n \longrightarrow P_\ell^n \longrightarrow W_\ell^n \longrightarrow 0,$$

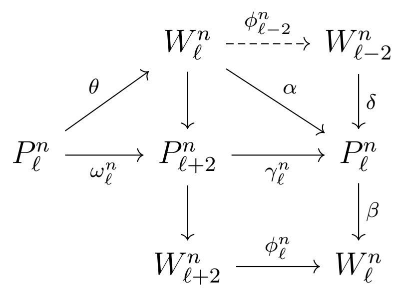
with the map from $W_{\ell-2}^n$ to P_ℓ^n injective and the map from P_ℓ^n to W_ℓ^n surjective. Hence, the map from P_ℓ^n to $P_{\ell+2}^n$ is characterized by the composition

$$P_\ell^n \longrightarrow W_\ell^n \longrightarrow P_{\ell+2}^n.$$

For the map from $P_{\ell+2}^n$ to P_ℓ^n, we may thus consider the following diagram.



Since the map from P_ℓ^n to W_ℓ^n is surjective, by projectivity it must factor through P_ℓ^n to obtain a nonzero map from $P_{\ell+2}^n$ to P_ℓ^n of the above form. Indeed, attaching in more short exact sequences, we may fold the above maps into the following diagram.



As all solid lines are known to commute, we note that $\beta \circ \alpha$ is exactly ϕ_ℓ^n composed with the map from W_ℓ^n to $W_{\ell+2}^n$ going down the middle column of the above diagram. However, this is exactly the zero map as the middle column is an exact sequence, implying that $\beta \circ \alpha = 0$. Additionally, since the rightmost column of the above diagram is another short exact sequence, it follows by Proposition A.1 that α factors through $W_{\ell-2}^n$ as $\alpha = \delta \circ \phi_{\ell-2}^n$. It follows that

$$\gamma_\ell^n \circ \omega_\ell^n = \alpha \circ \theta = \delta \circ \phi_{\ell-2}^n \circ \theta.$$

To show that $\gamma_\ell^n \circ \omega_\ell^n$ is nonzero, it suffices to show that $\phi_{\ell-2}^n$ is nonzero as δ is an inclusion while θ is a projection due to the construction of exact sequences. To avoid circularity, suppose we do not a priori know the construction of $\phi_{\ell-2}^n$. It is, however, easy to verify that it must be nonzero as by Corollary 4.8 it is simply given by the composition

$$W_\ell^n \longrightarrow L_\ell^n \longrightarrow W_{\ell-2}^n.$$

Additionally, we check easily that $\gamma_\ell^n \circ \omega_\ell^n$ is nonidentity, as otherwise this induces a section map on the P_ℓ^n with respect to the quotient W_ℓ^n, contradicting the fact that P_ℓ^n is indecomposable.

Now we consider the composition $\omega_\ell^n \circ \gamma_\ell^n$. This time, we have the following diagram.

$$\begin{array}{ccccc} P_{\ell+2}^n & \xrightarrow{\gamma_\ell^n} & P_\ell^n & \xrightarrow{\omega_\ell^n} & P_{\ell+2}^n \\ {\scriptstyle\mu}\downarrow & & \downarrow & \nearrow{\scriptstyle\nu} & \\ W_{\ell+2}^n & \xrightarrow{\phi_\ell^n} & W_\ell^n & & \end{array}$$

It is easy to check that everything commutes by projectivity, and in particular

$$\omega_\ell^n \circ \gamma_\ell^n = \nu \circ \phi_\ell^n \circ \mu.$$

Since ν is an inclusion while μ is a projection, it follows that $\omega_\ell^n \circ \gamma_\ell^n$ is nonzero and nonidentity for the exact same reasons as in the previous case. We are done. $\square$

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