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The Team

Arjun Agarwal (Student)
Jesuit High School
Portland, OR, U.S.A.

Dr. Arun Kannan (Supervisor)
MIT PRIMES
Chicago, IL, U.S.A.

Title

Semisimplifications of α_p-equivariant GL_n-modules

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SEMISIMPLIFICATIONS OF α_p-EQUIVARIANT GL_n-MODULES

ARJUN AGARWAL AND ARUN S. KANNAN

Abstract. We show that by using the semisimplification functor, one can explicitly construct restricted representations of $\mathfrak{gl}(m|n)$ from restricted representations of $\mathfrak{gl}(m+n(p-1))$. Therefore, by the Steinberg tensor product theorem, a solution to the character problem for $GL(m+n(p-1))$ implies a solution to that for $GL(m|n)$. Moreover, we show that the semisimplification of a simple module is semisimple and provide a method of decomposing those modules into simple modules based on highest-weight arguments. We also provide an algorithm which decomposes the highest weight module, $L(\lambda)$, into Jordan blocks to aid with calculations. Using these theorems and algorithm, we provide explicit calculations in low rank and characteristic.

Keywords: Representation theory, semisimplification, Lie superalgebras, restricted representations, Verlinde category, symmetric tensor category, highest weight modules, algorithm.

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Declaration of Academic Integrity

The participating team declares that the paper submitted is comprised of original research and results obtained under the guidance of the instructor. To the team's best knowledge, the paper does not contain research results, published or not, from a person who is not a team member, except for the content listed in the references and the acknowledgment. If there is any misinformation, we are willing to take all the related responsibilities.

Arjun Agarwal (Student): 

Dr. Arun Kannan (Instructor):

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X Arjun Agarwal
Name of team member: Arjun Agarwal

X
Name of supervising teacher: Dr. Arun Kannan

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1. Introduction

A long-standing problem in the representation theory of $GL(n)$ and other algebraic groups is to compute the characters of simple modules. In characteristic zero, this problem has been solved. In characteristic $p > 0$, on the other hand, this problem is notoriously difficult and has received great attention over the past fifty years.

Therefore, the same problem for algebraic supergroups like $G = GL(m|n)$ is even more difficult by virtue of being a generalization, and even in characteristic zero the full story is not known.

In positive characteristic, however, there is a silver lining. The Steinberg tensor product theorem (see [Ste63; Kuj06]) reduces the character problem for G to understanding that for the first Frobenius kernel $G_{(1)}$ of G. Since the distribution algebra of $G_{(1)}$ is finite-dimensional isomorphic restricted enveloping algebra of the Lie (super)algebra $\mathfrak{gl}(m|n)$ of G, this reduces the problem to studying the finitely-many simple restricted modules over $\mathfrak{gl}(m|n)$.

In general, however, this problem can be hard to tackle and so we need another lens to view these $\mathfrak{gl}(m|n)$ modules from. This is where semisimplification comes in; we can semisimplify $\mathfrak{gl}(m+n(p-1))$ restricted representations to obtain $\mathfrak{gl}(m|n)$ restricted representations.

More generally, consider a Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbb{K}$ with characteristic p. Consider a nilpotent derivation d of order at most p. This can be realized as a Lie algebra in the category Rep $\mathbb{K}[t]/(t^p)$ of $\mathbb{K}[t]/(t^p)$ by specializing d to t. The semisimplification of this category is the Verlinde category, Ver_p, which contains as a full subcategory the category of super vector spaces $\mathrm{sVec}_{\mathbb{K}}$. Therefore, the image $\overline{\mathfrak{g}}$ of $\mathfrak{g}$ under the semisimplification functor projected onto this full category is a Lie algebra in $\mathrm{sVec}_{\mathbb{K}}$, which is a Lie superalgebra (see [Kan22]).

1.1. **Outline of paper.** In §2 we review some basic theory of symmetric tensor categories and review additional categories relevant to this paper. We finish by proving a general theorem about the basis elements of the tensor of two indecomposable modules. The respresentation theory of Lie superalgebras is covered in §3. In §4 we review how Lie algebras behave under the semisimplification functor in $\operatorname{Rep}\boldsymbol{\alpha}_p$. We conclude by proving that the semisimplification of $L(\lambda)$ as a $\mathfrak{gl}(m+n(p-1))$ module is semisimple and also provide a method to decompose the semisimplified module into simple modules using highest-weight arguments. Finally, in §5 we provide a decomposition algorithm to decompose a highest weight module into its Jordan blocks. We end by computing these semisimplifications in low rank and characteristic. Finally, Appendix A summarizes the explicit action maps for the $\mathfrak{gl}(m|n)$ modules and the relevant Jordan blocks in the decomposition. The calculations in the appendix also verify the computational results obtained in Section §5.

2. Symmetric Tensor Category

Throughout this paper, we will assume that the reader has some basic understanding about symmetric tensor categories. Nonetheless, in this section we provide some exposition relevant to this paper. A quick reference is [EK23], and a more comprehensive reference is [EGNO]. In this paper we will utilize the notation in these references. In particular, for a symmetric tensor category $\mathcal{C}$, we will use $\oplus$ to denote direct sum, $\otimes$ to denote the monoidal product, * to denote the dual object, Hom to denote hom spaces, 1_X to denote the identity map on X, $\mathbb{1}$ to denote the unit object, and c, X, Y to denote the braiding map $X \otimes Y \to Y \otimes X$ for any two objects $X, Y \in \mathcal{C}$. We will always supress the associativity morphism from diagrams and equations.

Throughout this paper, $\mathbb{K}$ denotes an algebraically closed field of characteristic $p \geq 0$. We are usually interested when $p > 2$.

2.1. **Algebraic objects in symmetric tensor categories.** A symmetric tensor category can be thought of as a "home" to do algebra. For example, recall that a unital associative algebra A is a vector space with a bilinear map $m : A \times A \to A$ called multiplication such that $m(m(a, b), c) = m(a, m(b, c))$ (the associativity property) and such that there is an element $1 \in A$ such that $m(1, a) = m(a, 1) = a$ for all $a, b, c \in A$. Moreover, A is commutative if $m(a, b) = m(b, a)$ for all $a, b \in A$.

We can instead phrase the definition in the following, equivalent way. Let $\mathcal{C} = \mathrm{Vec}_{\mathbb{K}}$ denote the category of vector spaces (see §2.3.1). A unital associative algebra is an object $A \in \mathcal{C}$ with two maps $m : A \otimes A \to A$ and $\mu : \mathbb{1} \to A$ such that the following diagrams commute.

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}_A} & A \otimes A \\ {\scriptstyle \mathrm{id}_A \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array}$$

$$\begin{array}{ccccc} \mathbb{1}\otimes A & \xrightarrow{\mu\otimes \mathrm{id}_A} & A\otimes A & \xleftarrow{\mathrm{id}_A\otimes\mu} & A\otimes\mathbb{1} \\ & \searrow^{\cong} & \downarrow{\scriptstyle m} & \swarrow^{\cong} & \\ & & A & & \end{array}$$

Moreover, we can say A is commutative if $m \circ c_{A,A} = m$. The advantage of this formalism is that this definition extends to any symmetric tensor category. For instance, if $\mathcal{C}$ is the category of super vector spaces (see §2.3.2), we get the definition of a unital associative (super-commutative) superalgebra.

In this paper, we extend this viewpoint to Lie algebras. Recall that a Lie algebra over $\mathbb{K}$ is a vector space $\mathfrak{g}$ endowed with a $\mathbb{K}$-bilinear map $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is anti-symmetric (assuming char $\mathbb{K} \neq 2$) and satisfies the Jacobi identity. This can be phrased categorically as follows. The category $\mathrm{Vec}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$ is a symmetric tensor category endowed with the usual braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ given by interchanging X and Y, a natural isomorphism in objects X and Y. Then, a Lie algebra (in the category $\mathrm{Vec}_{\mathbb{K}}$) is an object $\mathfrak{g}$ equipped with a morphism $\beta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that the following relations of morphisms hold:

$$\beta \circ (1_{\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g},\mathfrak{g}}) = 0,$$
$$\beta \circ (\beta \otimes 1_{\mathfrak{g}}) \circ (1_{\mathfrak{g}^{\otimes 3}} + (123)_{\mathfrak{g}^{\otimes 3}} + (132)_{\mathfrak{g}^{\otimes 3}}) = 0,$$

where the permutation $(123)_{\mathfrak{g}^{\otimes 3}} : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}^{\otimes 3}$ is given by

$$(123)_{\mathfrak{g}^{\otimes 3}} \coloneqq (1_{\mathfrak{g}} \otimes c_{\mathfrak{g},\mathfrak{g}}) \circ (c_{\mathfrak{g},\mathfrak{g}} \otimes 1_{\mathfrak{g}}),$$

and the permutation $(132)_{\mathfrak{g}^{\otimes 3}} : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}^{\otimes 3}$ is given by

$$(132)_{\mathfrak{g}^{\otimes 3}} \coloneqq (c_{\mathfrak{g},\mathfrak{g}} \otimes 1_{\mathfrak{g}}) \circ (1_{\mathfrak{g}} \otimes c_{\mathfrak{g},\mathfrak{g}}).$$

The first relation corresponds to the anti-symmetry condition, and the second is the Jacobi identity. Using these as defining axioms, we can extend the definition to any symmetric tensor category $\mathcal{C}$ with braiding c, and call the pair $(\mathfrak{g}, \beta)$ an operadic Lie algebra in $\mathcal{C}$. A Lie algebra in a symmetric tensor category is an operadic Lie algebra with further imposed relations (see [EK23], [EGNO]) . All operadic Lie algebras we consider in this paper will satisfy these, so we will drop the adjective "operadic" from now on.

Given an object $V \in \mathcal{C}$, an example of an operadic Lie algebra is the *general linear Lie algebra* $\mathfrak{gl}(V) \coloneqq V \otimes V^*$, where the bracket β is given by

$$\beta = 1_V \otimes ev_V \otimes 1_{V^*} \circ (1_{\mathfrak{gl}(V)\otimes\mathfrak{gl}(V)} - c_{\mathfrak{gl}(V),\mathfrak{gl}(V)}),$$

where $ev_V : V^* \otimes V \to \mathbb{1}$ is the evaluation map and we implicitly apply the unit isomorphism.

This just a categorical rephrasing of the statement $[x, y] \coloneqq xy - yx$. Define the trace map $\mathrm{tr} : \mathfrak{gl}(V) \to \mathbb{1}$ to be the composition

$$\mathrm{tr} \coloneqq V \otimes V^* \xrightarrow{c_{V,V^*}} V^* \otimes V \xrightarrow{ev_V} \mathbb{1}.$$

It can be checked that tr is a Lie algebra homomorphism to the trivial Lie algebra, and therefore it's kernel is an ideal of $\mathfrak{gl}(V)$. This kernel is referred to as the *special linear Lie algebra* $\mathfrak{sl}(V)$.

One can continue in this vein to extend familiar notions to arbitrary symmetric tensor categories. For instance, a bilinear form $B : V \otimes V \to \mathbb{1}$ is *symmetric* if $B \circ c_{V,V} = B$ and is *skew-symmetric* if $B \circ c_{V,V} = -B$. It is *non-degenerate* if the induced map $V \to V^*$ is an isomorphism.

2.2. **Semisimplification.** Let $\mathcal{C}$ be a symmetric tensor category. For any objects $V, W \in \mathcal{C}$, a *negligible morphism* is a morphism $f : V \to W$ such that for all morphisms $g : W \to V$, the trace of the composition $f \circ g$ is zero. The collection of negligible morphisms in $\mathcal{C}$ form a tensor ideal. The symmetric tensor category obtained by quotienting out by the tensor ideal of negligible morphisms is called the *semisimplification* of $\mathcal{C}$, which will denote $\overline{\mathcal{C}}$.

Intuitively, the effect of this is forcing Schur's lemma to hold. In other words, the semisimplification of a symmetric tensor category is the symmetric tensor category obtained by declaring all indecomposable objects to be simple, except those whose categorical dimension is zero, which are sent to zero. We then define the tensor product the same way (for more details on semisimplification, see [EO21]).

The semisimplification is a semisimple symmetric tensor category by construction. There is a semisimplification functor from a symmetric tensor category $\mathcal{C}$ to its semisimplification $\overline{\mathcal{C}}$, and it is symmetric and monoidal. We will denote the images of objects and morphisms under this functor with an overline. While this functor is neither left nor right exact in general, it preserves isomorphisms and commutative diagrams, which means, for instance, that the semisimplification of an operadic Lie algebra is an operadic Lie algebra. Similarly, the semisimplification of a module over an operadic Lie algebra is a module over the semisimplification of that operadic Lie algebra.

2.3. **Relevant symmetric tensor categories.** In this subsection, we will describe the symmetric tensor categories that will be relevant to this paper.

2.3.1. *The category of vector spaces.* The category $\mathrm{Vec}_{\mathbb{K}}$ of vector spaces is a symmetric tensor category. The objects of this STC are vector spaces and morphisms are linear maps between vector spaces. The monoidal structure is given by usual tensor product. The braiding $c_{V,W}$ is given by $c_{V,W}(v \otimes w) = w \otimes v$ for all $v \in V$ and $w \in W$. Notice that this is the representation category of the trivial group.

2.3.2. *The category of super vector spaces.* The category $\mathrm{sVec}_{\mathbb{K}}$ is the symmetric tensor category whose objects are $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces and morphisms are gradation-preserving linear maps. In particular, we write a super vector space V as $V = V_{\overline{0}} \oplus V_{\overline{1}}$, and let $\operatorname{sdim} V = (\dim V_{\overline{0}} | \dim V_{\overline{1}})$. Here $\overline{0}, \overline{1} \in \mathbb{Z}/2\mathbb{Z}$ and distinguish the even and odd subspaces, respectively. Any vector lying solely in $V_{\overline{0}}$ or $V_{\overline{1}}$ is said to be *homogeneous*, and the parity function $|\cdot| : V_{\overline{0}} \cup V_{\overline{1}} \to \mathbb{Z}/2\mathbb{Z}$ assigns a homogeneous vector its *parity* based on whether it lies in the even or odd subspace. This category has a braiding $c_{V,W}$ given by the Koszul sign rule:

$$c_{V,W}(v \otimes w) = (-1)^{|v||w|}(w \otimes v), \tag{2.1}$$

where $v \in V$ and $w \in W$ are homogeneous.

An operadic Lie algebra in this category is simply a Lie superalgebra as we know it (outside characteristic $p = 2, 3$, where certain relations need to be imposed - this will not be a concern in this paper, as these relations are satisfied by the Lie algebras we will consider). In this paper, we will frequently be working with Lie superalgebras, so we point the reader to [CW12; Mus12]. Also notice that for any Lie superalgebra $\mathfrak{g}$ its even part $\mathfrak{g}_{\overline{0}}$ is an ordinary

Lie algebra (i.e. a Lie algebra in the category of vector spaces). From now on, when working with $\mathrm{Vec}_{\mathbb{K}}$ or $\mathrm{sVec}_{\mathbb{K}}$, we will drop the adjective "operadic" when discussing operadic Lie algebras.

Remark 2.1. Often times when working with the category of super vector spaces, we use the adjective "super" to distinguish between ordinary algebraic objects and their super analogs. In this paper, we will drop such adjectives because this is a benefit of working with symmetric tensor categories. For instance, rather than saying "super-commutative superalgebra", we can just say "commutative algebra in $\mathrm{sVec}_{\mathbb{K}}$". Similarly, a Lie algebra in $\mathrm{sVec}_{\mathbb{K}}$ is just a Lie superalgebra (modulo concerns about characteristics 2 and 3).

2.3.3. *The representation category* $\mathrm{Rep}\,\boldsymbol{\alpha}_p$. Let $\boldsymbol{\alpha}_p$ denote the kernel of the Frobenius endomorphism on the additive group scheme $\mathbb{G}_a$ over $\mathbb{K}$, whose characteristic p is strictly greater than 0. Its coordinate ring $\mathbb{K}\boldsymbol{\alpha}_p$ is $\mathbb{K}[t]/(t^p)$, which is a cocommutative Hopf algebra with comultiplication defined by letting t be primitive (this only works in characteristic p). The dual space $\mathbb{K}\boldsymbol{\alpha}_p^*$ of $\mathbb{K}\boldsymbol{\alpha}_p$ has basis given by $\{f_0, f_1, \ldots, f_{p-1}\}$, where $f_i(t^k) = \delta_{ik} i!$. The antipode is defined by sending $t \mapsto -t$. The comultiplication on $\mathbb{K}\boldsymbol{\alpha}_p$ gives a multiplication on $\mathbb{K}\boldsymbol{\alpha}_p^*$ where f_0 is the identity and $f_i f_j = f_{i+j}$ (let $f_i = 0$ for $i \geq p$). Therefore, as algebras, $\mathbb{K}\boldsymbol{\alpha}_p$ and $\mathbb{K}\boldsymbol{\alpha}_p^*$ are isomorphic under the map $t^i \mapsto f_i$. Because modules over the affine group scheme $\boldsymbol{\alpha}_p$ are determined by $\mathbb{K}\boldsymbol{\alpha}_p$-comodules, which themselves are $\mathbb{K}\boldsymbol{\alpha}_p^*$-modules, we will describe objects in the representation category $\mathrm{Rep}\,\boldsymbol{\alpha}_p$ of $\boldsymbol{\alpha}_p$ as finite-dimensional $\mathbb{K}[t]/(t^p)$-modules. For the remainder of this text, the symbol t will be used to refer to the corresponding element of $\mathbb{K}[t]/(t^p)$.

The category $\mathrm{Rep}\,\boldsymbol{\alpha}_p$ is a symmetric tensor category with braiding given by the usual braiding of vector spaces (there is a forgetful functor from $\mathrm{Rep}\,\boldsymbol{\alpha}_p$ to $\mathrm{Vec}_{\mathbb{K}}$). Hence, an example of a Lie algebra $(\mathfrak{g}, \beta)$ in $\mathrm{Rep}\,\boldsymbol{\alpha}_p$ is a Lie algebra in $\mathrm{Vec}_{\mathbb{K}}$ equipped with a nilpotent element $x \in \mathfrak{g}$ of order at most p; then $\mathfrak{g}$ is a $\mathbb{K}\boldsymbol{\alpha}_p$-module by letting t act as $\mathrm{ad}\, x$, and β is naturally a morphism in $\mathrm{Rep}\,\boldsymbol{\alpha}_p$ by the Jacobi identity (as a Lie algebra in $\mathrm{Vec}_{\mathbb{K}}$). More generally, we can take t to be any nilpotent derivation of order at most p (not necessarily inner).

The category $\mathrm{Rep}\,\boldsymbol{\alpha}_p$ is not semisimple; indeed, it contains non-simple indecomposable objects. The pairwise non-isomorphic indecomposable objects are given by the modules $J_n = \mathbb{K}^n$ where t acts as the nilpotent Jordan block of size n $(1 \leq n \leq p)$. If $v_1, v_2, \ldots, v_n$ is a basis of J_n such that $t \cdot v_i = v_{i+1}$, we will use the notation

$$v_1 \mapsto v_2 \mapsto \cdots \mapsto v_n$$

to refer to that particular object J_n. Given such a basis of J_n, it is a straightforward exercise to verify that

$$v_n^* \mapsto -v_{n-1}^* \mapsto \cdots \mapsto (-1)^{n-1} v_1^*$$

is a basis for J_n^*, where v_i^* is the dual basis defined by $v_i^*(v_j) = \delta_{ij}$.

The following theorem will be useful in doing calculations later on (theorems of this sort have been studied in the past; for instance, see [II06; Bar22]).

Theorem 2.2. *Suppose $V = J_{p-1}$ is an indecomposable module with basis $v_1 \mapsto \cdots \mapsto v_{p-1}$ and $W = J_k$ is an indecomposable module with basis $w_1 \mapsto \cdots \mapsto w_k$ with $1 \leq k \leq p-1$.*

Then, $V \otimes W = J_{p-k} \oplus (k-1)J_p$, where the J_{p-k} is generated by

$$\sum_{i=1}^{k} \binom{k}{i} v_i \otimes w_{k+1-i},$$

where we define $\binom{k}{i} = \frac{k(k-1)\cdots(k-i+1)}{i!}$.

Proof. We first show that there are $(k-1)$ linearly independent copies of J_p generated by $v_1 \otimes w_i$ for $1 \leq i < k$. Consider a vector of the form $v_1 \otimes w_i$ for $1 \leq i < k$. Applying successive powers of t yields

$$t^m(v_1 \otimes w_i) = \sum_{j=1}^{m+1} \binom{m}{j-1} v_j \otimes w_{i+m-j+1}.$$

For $m = p-1$ the term $v_{p-1} \otimes w_i$ appears with coefficient 1 (which does not vanish modulo p), and by linear independence cannot be canceled by other terms in the summation. Therefore, $t^{p-1}(v_1 \otimes w_i) \neq 0$. For $m = p$, the binomial coefficients $\binom{p}{j}$ vanish modulo p except when $j = 0$ or $j = p$, but in those cases either the v- or w-component is zero. Thus $t^p(v_1 \otimes w_i) = 0$. Therefore, each $v_1 \otimes w_i$ generates a copy of J_p, yielding $(k-1)$ copies of J_p inside $V \otimes W$.

Now consider,

$$u = \sum_{i=1}^{k} \binom{k}{i} v_i \otimes w_{k+1-i}.$$

We claim that the submodule generated by u is isomorphic to J_{p-k}. Indeed, applying t^j to u gives

$$t^j u = \sum_{i=1}^{k} \binom{k+j}{i+j} v_{i+j} \otimes w_{k+1-i}.$$

For $j = p-k$, this becomes

$$t^{p-k} u = \sum_{i=1}^{k} \binom{p}{p-k+i} v_{i+p-k} \otimes w_{k+1-i}.$$

All binomial coefficients vanish modulo p except when $i = k$, in which case the term remaining is $v_p \otimes w_1 = 0$. Hence $t^{p-k}u = 0$. On the other hand, $t^{p-k-1}u \neq 0$ because the surviving terms involve $v_{p-1} \otimes w_1$, which is nonzero. Thus u generates a Jordan block of length $p-k$.

Additionally, the vector u is linearly independent from vectors generated in the $(k-1)$ copies of J_p. To see this, note that if u were a linear combination of $t^{k-i}(v_1 \otimes w_i)$ for $1 \leq i < k$, then applying t^{p-k} would yield a nontrivial linear relation among $\{t^{p-i}(v_1 \otimes w_i)\}_{i=1}^{k-1}$, contradicting the linear independence of these vectors. Therefore u lies outside the span of the J_p summands, and their intersection is trivial.

□

Often times we will have a vector space V and need to view it as an object in $\operatorname{Rep} \boldsymbol{\alpha}_p$ with respect to two different t-actions. Therefore, if $x : V \to V$ is a nilpotent endomorphism with $x^{[p]} = 0$ (p-fold composition), then we will use the notation (V, x) to refer to the object in $\operatorname{Rep} \boldsymbol{\alpha}_p$ where t acts on V as x.

2.3.4. *The Verlinde Category.* The Verlinde category Ver_p is by definition the semisimplification of $\operatorname{Rep} \boldsymbol{\alpha}_p$. Therefore, the simple objects in Ver_p are $L_1, \ldots, L_{p-1}$, which are the images of $J_1, \ldots, J_{p-1}$ under the semisimplification functor, respectively, i.e. $L_i = \overline{J_i}$. If $v_1 \mapsto v_2 \mapsto \cdots \mapsto v_i$ denotes a J_i, we will refer to the corresponding copy of L_i by $\overline{v_1 \mapsto v_2 \mapsto \cdots \mapsto v_i}$ (for $i < p$). Note that J_p is sent to the zero object as it is p-dimensional, so its categorical dimension is 0. In terms of negligible morphisms, this is because any sequence of morphisms $J_i \to J_p \to J_i$ and $J_p \to J_i \to J_p$ for any i has trace zero, so in the semisimplification there are no nonzero morphisms in or out of the image of J_p, meaning its image is zero. More generally, any map $J_i \to J_k$ is negligible unless it's an isomorphism.

It is well known that the tensor product is given by the truncated Clebsch-Gordan rule (see [Ost20]), which is similar to the usual Clebsch-Gordan rule of $\mathfrak{sl}(2, \mathbb{C})$-modules (the truncation comes from the terms in bold):

$$L_m \otimes L_n = \bigoplus_{i=1}^{\min(m,n,\boldsymbol{p-m},\boldsymbol{p-n})} L_{|m-n|+2i-1}. \tag{2.2}$$

In particular, $\mathbb{1} := L_1$ is the unit object with respect to tensor product. More importantly, we have the following proposition:

Proposition 2.3. *The category* $\mathrm{sVec}_{\mathbb{K}}$ *is symmetric tensor equivalent to the subcategory generated by the objects* L_1 *and* L_{p-1} *in* Ver_p.

Proof. This is well-known; see the proof of Proposition 3.2.1 in [Kan22] for details. That proof, however, mainly relies on proving the fact that $J_{p-1} \otimes J_{p-1} = J_1 \oplus (p-2)J_{p-1}$ and producing a spanning vector for the J_1. This is just a specific case of the proof of Theorem 2.2 in this paper. In particular, notice that $\binom{p-1}{i} = (-1)^i$ in characteristic p. ☐

Notice that tensoring with L_{p-1} is an autoequivalence on Ver_p that sends L_i to L_{p-i}. We will call this functor the *parity shift functor*, motivated by the observation that there is an "even" sub-symmetric tensor category Ver_p^+ spanned by the objects $L_1, L_3, \ldots, L_{p-2}$ and a decomposition $\mathrm{Ver}_p = \mathrm{Ver}_p^+ \boxtimes \mathrm{sVec}_{\mathbb{K}}$, where $\boxtimes$ denotes the Deligne tensor product. The following is a consequence of Proposition 2.3:

Corollary 2.4. *Let* $V = n_1 J_1 \oplus n_{p-1} J_{p-1}$ *be an object in* $\mathrm{Rep}\, \boldsymbol{\alpha}_p$. *Then, its semisimplification is a super vector space.*

3. The Lie superalgebra $\mathfrak{gl}(m|n)$

The primary goal of this paper is study the images of simple $GL(m + n(p-1))$-modules under the semisimplifcation functor and understand the resulting $GL(m|n)$-modules. The Steinberg tensor product theorem lets us relate simple $GL(m|n)$-modules to those of the first Frobenius kernel $GL(m|n)_{(1)}$ of $GL(m|n)$. In turn, the representation theory of $GL(m|n)_{(1)}$ is equivalent to the restricted representation theory of its Lie algebra, which is $\mathfrak{gl}(m|n)$. Therefore, we will work with the Lie algebra $\mathfrak{gl}(m + n(p-1))$ and the Lie superalgebra $\mathfrak{gl}(m|n)$, which is much simpler, from both a technical and expository perspective.

3.1. **Definitions.** Let $\mathbb{K}^{m|n}$ denote the $(m|n)$-dimensional super vector space of column vectors with entries in $\mathbb{K}$, where column vectors whose last n entries are zero are homogeneous and even and column vectors whose first m entries are zero are homogeneous and odd. We will index the rows by the indexing set $\mathbb{I} = \{1, 2, \ldots, m, \overline{1}, \overline{2}, \ldots, \overline{n}\}$, in that order. It will also be convenient when indexing to let $\overline{i} = m + i$. We assign a parity function on $\mathbb{I}$, also denoted $|\cdot|$, by $|i| = 0$ if $i \in \{1, \ldots, m\}$ and $|i| = 1$ if $i \in \{\overline{1}, \ldots, \overline{n}\}$

Let e_i be the column vector with a 1 in the i-th spot and zero elsewhere. Then $\{e_i\}_{i\in\mathbb{I}}$ is a basis for $\mathbb{K}^{m|n}$. We will let $\{e_i^*\}_{i\in\mathbb{I}}$ denote the dual basis. These give a basis

$$\{e_{ij} := e_i \otimes e_j^*\}_{i,j\in\mathbb{I}}$$

of $\mathrm{End}(\mathbb{K}^{m|n}) = \mathbb{K}^{m|n} \otimes (\mathbb{K}^{m|n})^*$. The space can then be identified with the space of $(m+n)\times(m+n)$ matrices of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{3.1}$$

where A is $m\times m$, B is $m\times n$, C is $n\times m$, and D is $n\times n$. Such matrices with $B, C = 0$ are purely even, and those with $A, D = 0$ are purely odd. To emphasize this gradation, we call this an $(m|n)\times(m|n)$ matrix. In particular, e_{ij} is the elementary matrix with 1 in the (i,j)-entry and zero elsewhere. Notice that the parity of e_{ij} is $|i|+|j|$, so even if i, j have the same parity and odd if they have different parity.

Definition 3.1. The Lie superalgebra $\mathfrak{gl}(m|n)$ is the Lie superalgebra given by endowing the space of $(m|n)\times(m|n)$-matrices with the bracket

$$[x,y] = xy - (-1)^{|x||y|}yx.$$

The key distinction is that $[x,y] = xy + yx$ when both x, y are odd. We will simply write $\mathfrak{gl}(m)$ to refer to $\mathfrak{gl}(m|0)$. Given a matrix $x \in \mathfrak{gl}(m|n)$ as in 3.1, we define the *supertrace* $\mathrm{str} : \mathfrak{gl}(m|n) \to \mathbb{K}$ to be the map

$$\mathrm{str}(x) = \mathrm{tr}(A) - \mathrm{tr}(D).$$

This is a generalization of the trace and can be checked that this is a Lie superalgebra homomorphism, where $\mathbb{K}$ is the trivial Lie algebra in purely even degree. The *special linear Lie superalgebra* is defined by $\mathfrak{sl}(m|n) := \ker \mathrm{str}$.

It will be useful to define a map called the *super transpose* from $\mathfrak{gl}(m|n)$ to itself, a generalization of the transpose. We denote the super tranpose with ST in the superscript, and for $x \in \mathfrak{gl}(m|n)$ in block form as in 3.1, it is defined by

$$x^{ST} := \left[\begin{array}{c|c} A^T & C^T \\ \hline -B^T & D^T \end{array}\right],$$

where A^T is the ordinary transpose of matrix A. It can be checked that the map $\theta(x) = -x^{ST}$ is a Lie superalgebra automorphism, which is called the *Chevalley automorphism*.

Like the ordinary Lie algebra $\mathfrak{gl}(m)$, the Lie superalgebra $\mathfrak{gl}(m|n)$ admits a triangular decomposition and root system. Let $\mathfrak{h}$ denote the subalgebra of diagonal matrices in $\mathfrak{gl}(m|n)$, $\mathfrak{n}^+$ the strictly upper-triangular matrices, and $\mathfrak{n}^-$ the strictly lower triangular matrices. A basis for $\mathfrak{h}$ is given by $\{e_{ii}\}_{i\in\mathbb{I}}$. Let $\{\epsilon_i\}_{i\in\mathbb{I}}$ be the dual basis of $\mathfrak{h}^*$. Given an element $\lambda = \sum_{i\in\mathbb{I}} \lambda_i\epsilon_i \in \mathfrak{h}^*$, we will often use the notation $(\lambda_1, \dots \lambda_m, \lambda_{\overline{1}}, \dots \lambda_{\overline{n}})$ to refer to it. Notice that for any $h \in \mathfrak{h}$, we have $[h, e_{ij}] = (\epsilon_i - \epsilon_j)(h)e_{ij}$, and therefore the roots of $\mathfrak{gl}(m|n)$ are $\{\epsilon_i - \epsilon_j\}_{i\neq j\in\mathbb{I}}$.

3.2. **Representation theory.** In this subsection, we discuss the restricted representation theory of $\mathfrak{gl}(m|n)$. In §3.2.1, we construct the simple restricted representations of $\mathfrak{gl}(m|n)$, and in §3.2.2 we explain how this relates to the representation theory of the affine group scheme $GL(m|n)$.

3.2.1. *Restricted representations.* Given a matrix $x \in \mathfrak{gl}(m|n)_{\bar{0}}$, let $x^{[p]}$ denote its p-th power. Let $\mathfrak{g}$ denote a Lie subalgebra of $\mathfrak{gl}(m|n)$ such that $\mathfrak{g}_{\bar{0}}$ is closed under the $[p]$ operation, and let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. We define the *restricted universal enveloping algebra* $u(\mathfrak{g})$ to be

$$u(\mathfrak{g}) := U(\mathfrak{g})/\langle x^p - x^{[p]}, x \in \mathfrak{g}_{\bar{0}}\rangle,$$

where $x^{[p]}$ denote the usual p-th power of a matrix $x \in \mathfrak{g}_{\bar{0}}$. A representation of $u(\mathfrak{g})$ is called a *restricted representation* or *restricted module* (with character $\chi = 0$) of $\mathfrak{g}$. We only concern ourselves with restricted modules, so we will drop the adjective "restricted" hence forth.

To construct the simple modules of $\mathfrak{gl}(m|n)$, we can use a Verma module argument. Define the set of weights $\mathfrak{h}_p^*$ by the collection of $\lambda \in \mathfrak{h}^*$ such that when written $\lambda = \sum_{i\in\mathbb{I}} \lambda_i \epsilon_i$, we have $\lambda_i \in \mathbb{Z}/p\mathbb{Z} \subset \mathbb{K}$. Then, $\lambda \in \mathfrak{h}^*$ is in $\mathfrak{h}_p^*$ if and only if for any $h \in \mathfrak{h}$, $\lambda(h^{[p]}) = \lambda(h)^p$. Therefore, we have a one-dimensional $\mathfrak{h}$-module $\mathbb{K}_\lambda$ spanned by a vector v such that $h.v = \lambda(h)v$. Then, we can extend this to a module over $\mathfrak{h} \oplus \mathfrak{n}^+$ by letting $\mathfrak{n}^+$ act trivially. Any vector $\mathfrak{n}^+$ annihilates will be called a *highest weight vector*. Finally, we define the baby Verma module

$$M(\lambda) := u(\mathfrak{gl}(m|n)) \otimes_{u(\mathfrak{h}\oplus\mathfrak{n}^+)} \mathbb{K}_\lambda,$$

which has the usual universal property. We define $L(\lambda)$ to be the quotient of $M(\lambda)$ by its unique maximal proper submodule. It is clear the collection $\{L(\lambda)\}_{\lambda\in\mathfrak{h}_p^*}$ is pairwise nonisomorphic and exhausts the simple $u(\mathfrak{g})$-modules (see [Shu24]).

3.2.2. *Relationship to $GL(m|n)$ representation theory.* The affine group scheme $GL(m|n)$ can be defined in the same way as the affine group scheme $GL(m)$ except by changing the underlying category from $\mathrm{Vec}_{\mathbb{K}}$ to $\mathrm{sVec}_{\mathbb{K}}$ and the defining object from $\mathbb{K}^m$ to $\mathbb{K}^{m|n}$. Its Lie algebra is $\mathfrak{gl}(m|n)$.

By the theory of Harish-Chandra pairs (see [Mas12]), there is a parabolic subgroup P of $GL(m|n)$ whose even subgroup is $GL(m|n)_{\bar{0}} = GL(m) \times GL(n)$ and whose Lie algebra is $\mathfrak{p} := \mathfrak{gl}(m|n)_{\bar{0}} + \mathfrak{n}^+$. Any simple module M over $GL(m|n)_{\bar{0}}$ can be trivially extended to a simple module over P. This module can then be induced to the entirety of $GL(m|n)$ to construct a *Kac module*, which has a unique simple quotient. Because the distribution algebra of $GL(m|n)$ is finite over that of P, all Kac modules are finite-dimensional, and therefore the simple $GL(m|n)$-modules are in bijection with those of the underlying even subgroup $GL(m|n)_{\bar{0}}$.

It is well known (see [Jan03]) that the simple $GL(m|n)_{\bar{0}}$-modules are canonically in bijection with the set of dominant integral weights, which we can identify with the set

$$X(T)^+ := \{\lambda \in \mathbb{Z}^{m+n} \mid \lambda_1 \geq \cdots \geq \lambda_m, \lambda_{m+1} \geq \cdots \geq \lambda_{m+n}\},$$

and we will denote the corresponding simple $GL(m|n)$-module as $\mathcal{L}(\lambda)$. Call $\lambda \in X(T)^+$ *restricted* if the difference between adjacent entries is at most $p-1$, except between λ_m and λ_{m+1}, where no condition is imposed. Then, any dominant integral weight λ can be written as $\nu + p\mu$, where ν is restricted and μ is also dominant integral. The Steinberg tensor product theorem (see [Kuj06]), tells us that $\mathcal{L}(\lambda) \cong \mathcal{L}(\nu) \otimes F^*\mathcal{L}_0(\mu)$, where $\mathcal{L}_0(\mu)$ is the simple $GL(m|n)_{\bar{0}}$-module of highest weight μ and F^* denotes the Frobenius twist of a module.

The point is that by decomposing $\lambda \in X(T)^+$ p-adically and iteratively applying this theorem, the character problem reduces to understanding the character problem for $\mathcal{L}(\lambda)$

for restricted λ. For such λ, the restriction of this module to the first Frobenius kernel $GL(m|n)_{(1)}$ of $GL(m|n)$ remains simple. And since the distribution algebra of $GL(m|n)_{(1)}$ is isomorphic as Hopf algebras to $u(\mathfrak{gl}(m|n))$, we reduce the character problem to that for simple restricted modules over $\mathfrak{gl}(m|n)$.

Moreover, the modules are identified in an obvious way. Specifically, recall we defined the $\mathfrak{gl}(m|n)$-module $L(\lambda)$ for $\lambda \in \mathfrak{h}_p^*$ using the standard triangular decomposition on $\mathfrak{gl}(m|n)$ and standard basis for $\mathfrak{h}_p^*$. This gives a triangular decomposition on $GL(m|n)$, and this standard basis comes from differentiating the obvious coordinate maps $H \to \mathbb{G}_m$, where H is the subgroup of diagonal matrices in $GL(m|n)_{\bar{0}}$. If we define the $GL(m|n)$-module $\mathcal{L}(\lambda)$ for $\lambda \in X(T)^+$ with respect to these choices, then, for restricted λ, we have

$$L(\lambda_1\epsilon_1 + \cdots \lambda_{m+n}\epsilon_{m+n}) \cong \mathcal{L}(\lambda)$$

as $\mathfrak{gl}(m|n)$-modules, where on the left-hand side each λ_i is projected to $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{K}$, and where on the right-hand side the module structure is given by restriction.

3.2.3. *The Shapovalov form.* Let V be $\mathfrak{gl}(m|n)$-module. A bilinear form $\beta : V \otimes V \to \mathbb{1}$ on V is said to be *contravariant* if for $x \in \mathfrak{gl}(m|n)$ and $v, w \in V$, we have $\beta(x.v \otimes w) + \beta(v \otimes \theta(x).w) = 0$, where θ is the Chevalley automorphism. Now, suppose that V is a rational (i.e. one that lifts to $GL(m|n)$) highest weight module generated by the vector v. It can be show that V admits a unique (up to scaling) bilinear form S called the *Shapovalov form* such that $S(v \otimes v) = 1$ (see [Sha79] for more details). Moreover, it can be shown that S is symmetric and that the radical of this form is the unique maximal submodule of V. In particular, each simple module has a non-degenerate Shapovalov form. The following proposition is easy to see.

Proposition 3.1. *Let V be a $\mathfrak{gl}(m|n)$-module with a non-degenerate symmetric contravariant form β. Then V is semisimple.*

Proof. Let W be a simple submodule in V, and let $W^\perp$ be its orthogonal complement with respect to β. Clearly $W^\perp$ is a submodule as well. The claim follows by induction. □

3.2.4. *Explicit description of simple $\mathfrak{gl}(1|1)$-modules.* Here we explicitly describe the simple $\mathfrak{gl}(1|1)$-modules. For $\lambda = (\lambda_1, \lambda_{\bar{1}}) \in \mathfrak{h}_p^*$, the baby Verma module $M(\lambda)$ is $(1|1)$-dimensional, spanned by the vectors $1 \otimes v$ and $e_{\bar{1}1} \otimes v$. Notice that

$$e_{1\bar{1}}.(e_{\bar{1}1} \otimes v) = (-e_{\bar{1}1}e_{1\bar{1}} + (e_{11} + e_{\bar{1}\bar{1}})) \otimes v = (\lambda_1 + \lambda_{\bar{1}})(1 \otimes v).$$

Therefore, $L(\lambda)$ is $(1|1)$-dimensional and simple when $\lambda_1 + \lambda_{\bar{1}} \neq 0$ and 1-dimensional and simple when $\lambda_1 + \lambda_{\bar{1}} = 0$ (up to parity shift). These remain simple upon restriction to $\mathfrak{sl}(1|1)$, and are isomorphic as $\mathfrak{sl}(1|1)$-modules for fixed values of $\lambda_1 + \lambda_{\bar{1}}$ because the $\mathfrak{sl}(1|1)$ action cannot detect the constant $\lambda_1 - \lambda_{\bar{1}}$.

3.2.5. *Explicit description of simple $\mathfrak{gl}(2|1)$-modules.* The Lie algebra $\mathfrak{gl}(2|1) = \mathfrak{sl}(2|1) \oplus \mathbb{K}I_{m|n}$, where $I_{m|n}$ is the identity matrix. Therefore, by Schur's lemma, any simple $\mathfrak{gl}(2|1)$-module is given by a simple $\mathfrak{sl}(2|1)$-module and a scalar in $\mathbb{Z}/p\mathbb{Z}$ specifying the action of the center. Simple $\mathfrak{sl}(2|1)$-modules are described in [Zha09].

Let $\lambda = (\lambda_1, \lambda_2, \lambda_{\bar{1}}) \in \mathfrak{h}_p^*$. Notice there is a copy of $\mathfrak{gl}(2)$ in $\mathfrak{gl}(2|1)$ spanned by e_{11}, e_{12}, e_{21} and e_{22}. Let us describe its simple modules, which is classically known and will be stated without proof. Let l be the integer such that $0 \leq l < p$ and $(\lambda_1 - \lambda_2) \equiv l \pmod p$. Then, the vector space

$$L_0'(\lambda_1, \lambda_2) = \text{span}\{v, e_{21}v, \ldots, e_{21}^l v\}$$

uniquely admits the structure of a $\mathfrak{gl}(2)$-module where each basis vector is a weight vector, v is a highest vector with weight (λ_1, λ_2), and e_{21} acts in the obvious way. Moreover, $L_0'(\lambda_1, \lambda_2)$ is a simple $\mathfrak{gl}(2)$-module and all such modules arise in this fashion.

By letting $e_{\bar{1}\bar{1}}.(e_{21}^k v) = \lambda_{\bar{1}} e_{21}^k v$, the module $L_0'(\lambda_1, \lambda_2)$ becomes a $\mathfrak{gl}(2|1)_{\bar{0}}$-module which we'll call $L_0(\lambda_1, \lambda_2, \lambda_{\bar{1}})$. By extending trivially, this becomes a $\mathfrak{gl}(2|1)_{\bar{0}} + \mathfrak{n}^+$-module, and finally we define the Kac module

$$K(\lambda_1, \lambda_2, \lambda_{\bar{1}}) = u(\mathfrak{gl}(m|n)) \otimes_{u(\mathfrak{gl}(2|1)_{\bar{0}} + \mathfrak{n}^+)} L_0(\lambda_1, \lambda_2, \lambda_{\bar{1}}).$$

This module has a basis $\{e_{\bar{1}1}^i e_{\bar{1}2}^j \otimes e_{21}^k\}$ with $0 \le i, j \le 1$ and $0 \le k \le l$ and is $(2l+2|2l+2)$-dimensional. By the universal property of the baby Verma module, there are surjections $M(\lambda) \twoheadrightarrow K(\lambda) \twoheadrightarrow L(\lambda)$.

Now, we will state the results (rephrased for our setting) in [Zha09] to describe the structure of $L(\lambda)$. Let $\eta_i = \lambda_i + \lambda_{\bar{1}}$ and for $i = 1, 2$. By Proposition 3.1 in [Zha09], $L(\lambda) = K(\lambda)$ is simple if $\eta_1 \neq -1$ and $\eta_2 \neq 0$. The remaining cases are handled by the following:

Theorem 3.2 (Theorem 3.7 in [Zha09]). *Let J denote the unique maximal proper submodule in $K(\lambda)$. Then,*

(1) if $\eta_1 = -1$ and $\eta_2 \neq -1, 0$, then J is generated by the maximal vector $e_{\bar{1}1} \otimes v - (\eta_2 + 1)^{-1} e_{\bar{1}2} \otimes e_{21} v$ and $\dim L(\lambda) = 2(l+1) + 1$;
(2) if $\eta_1 = -1$ and $\eta_2 = 0$, then J is generated by $e_{\bar{1}2} \otimes v$ and $\dim L(\lambda) = 2p$ [1];
(3) if $\eta_1 = -1$ and $\eta_2 = -1$, then $J = \mathbb{K} e_{\bar{1}1} e_{\bar{1}2} \otimes v$ and $\dim L(\lambda) = 3$;
(4) if $\eta_1 \neq -1$ and $\eta_2 = 0$, then J is generated by $e_{\bar{1}2} \otimes v$ and $\dim L(\lambda) = 2\eta_1 + 1$ where η_1 is interpreted as the smallest nonnegative integer whose residue class mod p is η_1.

If λ is changed so that η_1 and η_2 do not change, then $L(\lambda)$ is unchanged as an $\mathfrak{sl}(2|1)$-module.

4. Applications of the semisimplification functor

In this section, we see how general linear Lie algebras behave under the semisimplification functor in $\operatorname{Rep} \boldsymbol{\alpha}_p$.

4.1. General considerations.

Let $\mathcal{C}$ be a symmetric tensor category and $V \in \mathcal{C}$ and object. Then, we have the following proposition:

Proposition 4.1. *The semisimplification $\overline{\mathfrak{gl}(V)}$ is isomorphic as Lie algebras to $\mathfrak{gl}(\overline{V})$ in $\mathcal{C}$ (resp. $\overline{\mathfrak{sl}(V)}$ and $\mathfrak{sl}(\overline{V})$).*

Proof. This is a consequence of the fact that the semisimplification functor is symmetric monoidal. □

[1] If there is no transcription error, then we believe this is a misprint in the original reference and maybe should say $2p-1$. For instance, consider the natural representation $\mathbb{K}^{2|1}$ of $\mathfrak{gl}(2|1)$, which has highest weight $(1, 0, 0)$ and basis $e_1, e_2, e_{\bar{1}}$. In characteristic $p \neq 2$, the symmetric square $S^2(\mathbb{K}^{2|1})$ is simple and has highest weight $(2, 0, 0)$. It is 5-dimensional, with basis $e_1^2, e_2^2, e_1e_2, e_1e_3, e_2e_3$. As an $\mathfrak{sl}(2|1)$-module in characteristic 3, $\eta_1 = 2 = -1$ and $\eta_2 = 0$, from which the theorem would suggest that this module is $2(3) = 6$ dimensional.

Now, suppose that $\mathcal{C} = \operatorname{Rep} \boldsymbol{\alpha}_p$, and let $\operatorname{Rep}_{\mathcal{C}}(\mathfrak{gl}(n), T)$ denote the (symmetric tensor) category whose objects are $\mathfrak{gl}(n)$-modules V equipped with a t-action given an operator T' such that the module map $(\mathfrak{gl}(n), T) \otimes (V, T') \to (V, T')$ is $\boldsymbol{\alpha}_p$-equivariant and whose morphisms are $\mathfrak{gl}(n)$-module maps that are also $\boldsymbol{\alpha}_p$-equivariant. We will want to study this category, because the semisimplification functor $\operatorname{Rep} \boldsymbol{\alpha}_p \to \operatorname{Ver}_p$ induces a functor $\operatorname{Rep}_{\mathcal{C}}(\mathfrak{gl}(n), T) \to \operatorname{Rep}_{\operatorname{Ver}_p} \overline{(\mathfrak{gl}(n), T)}$, where the target category is the representation category of $\overline{(\mathfrak{gl}(n), T)}$ as a Lie algebra in Ver_p. We have the following proposition:

Proposition 4.2. *Let $\rho : \mathfrak{gl}(n) \to \mathfrak{gl}(V)$ be a $\mathfrak{gl}(n)$-representation (not necessarily restricted), and let $x \in \mathfrak{gl}(n)$ satisfy $x^{[p]} = 0$ and $\rho(x)^{[p]} = 0$. Then, $(V, \rho(x))$ is an object in $\operatorname{Rep}_{\mathcal{C}}(\mathfrak{gl}(n), \operatorname{ad} x)$.*

Proof. Let ρ also denote the module map $\mathfrak{gl}(n) \otimes V \to V$. Then, for any $y \in \mathfrak{gl}(n)$ and $v \in V$, we have

$$\rho(t.(y \otimes V)) = \rho([x, y] \otimes v + y \otimes x.v) = [x, y].v + y.x.v = x.y.v = t.\rho(y \otimes v).$$

This shows that the module map is also $\boldsymbol{\alpha}_p$-equivariant. □

Corollary 4.3. *If λ is a restricted dominant integral weight for $\mathfrak{gl}(n)$ with associated simple representation $\rho : \mathfrak{gl}(n) \to \mathfrak{gl}(L(\lambda))$ and $x \in \mathfrak{gl}(n)$ is such that $x^{[p]} = 0$, then $(L(\lambda), \rho(x))$ is an object in $\operatorname{Rep}_{\mathcal{C}}(\mathfrak{gl}(n), \operatorname{ad} x)$.*

Proof. The representation factors through $u(\mathfrak{gl}(n))$, where $x^p = x^{[p]} = 0$ by the defining relations of the restricted enveloping algebra. This means $\rho(x)^{[p]} = 0$ as well. □

For the following two propositions, suppose $x, y \in \mathfrak{gl}(n)$ are such that $x^{[p]} = y^{[p]} = 0$ and are in the same nilpotent orbit with respect to the conjugation action of GL_n on $\mathfrak{gl}(n)$ i.e. there is a $g \in GL_n$ such that $gxg^{-1} = y$.

Proposition 4.4. *The Lie algebras $(\mathfrak{gl}(n), \operatorname{ad} x)$ and $(\mathfrak{gl}(n), \operatorname{ad} y)$ in $\operatorname{Rep} \boldsymbol{\alpha}_p$ are isomorphic. Therefore, their semisimplifications are isomorphic as Lie algebras in Ver_p.*

Proof. Define the map $\iota : (\mathfrak{gl}(n), \operatorname{ad} x) \to (\mathfrak{gl}(n), \operatorname{ad} y)$ by $z \mapsto gzg^{-1}$. Clearly, ι is a bijection, so we just need to show it is a Lie algebra homomorphism and a $\boldsymbol{\alpha}_p$-equivariant map. For any $w, z \in (\mathfrak{gl}(n), \operatorname{ad} x)$, we have

$$\iota([w, z]) = \iota(wz - zw) = g(wz - zw)g^{-1} = (gwg^{-1})(gzg^{-1}) - (gzg^{-1})(gwg^{-1}) = [\iota(z), \iota(w)],$$

so it is a Lie algebra homomorphism. Moreover, we have

$$t.\iota(z) = [y, gzg^{-1}] = [gxg^{-1}, gzg^{-1}] = [\iota(x), \iota(z)] = \iota([x, z]) = \iota(t.z).$$

Therefore, it ι is a Lie algebra isomorphism in $\operatorname{Rep} \boldsymbol{\alpha}_p$. The statement about an isomorphism of Lie algebras in Ver_p follows from the fact that the semisimplification functor is symmetric monoidal. □

This means that if $\rho : V \to V$ is a $\mathfrak{gl}(n)$-representation, then $(V, \rho(y))$ is a module over $(\mathfrak{gl}(n), \operatorname{ad} x)$ via the composite map

$$(\mathfrak{gl}(n), \operatorname{ad} x) \otimes (V, \rho(y)) \xrightarrow{\iota \otimes 1_{(V, \rho(y))}} (\mathfrak{gl}(n), \operatorname{ad} y) \otimes (V, \rho(y)) \xrightarrow{\rho} (V, \rho(y)).$$

We also get an equivalence of categories between $\operatorname{Rep}_{\mathcal{C}}(\mathfrak{gl}(n), \operatorname{ad} x)$ and $\operatorname{Rep}_{\mathcal{C}}(\mathfrak{gl}(n), \operatorname{ad} x)$.

Proposition 4.5. *Let $\rho : GL_n \to GL(V)$ be a rational representation of GL_n. Then, the semisimplifications $\overline{(V, d\rho(x))}$ and $\overline{(V, d\rho(y))}$ are isomorphic as $\overline{(\mathfrak{gl}(n), \operatorname{ad} x)}$ modules.*

Proof. Given a group G, let $c_h : G \to G$ denote conjugation by an element $h \in G$ (the group G will be clear from context). Now, for all $k \in GL_n$, we have

$$\rho(g)^{-1}\rho(gkg^{-1})\rho(g) = \rho(g)$$

which can be phrased as the following equality of maps:

$$c_{\rho(g)^{-1}} \circ \rho \circ c_g = \rho.$$

Differentiating this map at the identity tells us that

$$(dc_{\rho(g)^{-1}}) \circ d\rho \circ dc_g = d\rho,$$

which means that for all $z \in \mathfrak{gl}(n)$, we have

$$\rho(g)^{-1}d\rho(gzg^{-1})\rho(g) = d\rho(z).$$

Now, define the map $\theta : (V, x) \to (V, y)$ by $v \mapsto \rho(g)v$. This is obviously a bijection. We first claim that this is a Lie algebra module homomorphism. We have for all $z \in \mathfrak{gl}(n)$ and $z \in V$,

$$\theta(z.v) = \rho(g)d\rho(z)v = d\rho(gzg^{-1})\rho(g)v = (d\rho \circ \iota)(z)\theta(v) = z.\theta(v).$$

This calculation also shows that this homomorphism is $\boldsymbol{\alpha}_p$-equivariant, as

$$\theta(t.v) = \theta(x.v) = x.\theta(v) = t.\theta(v)$$

as the second x-action is essentially the original y-action due to the isomorphism ι. The proposition follows from Proposition 4.4 and the semisimplification functor being symmetric monoidal. □

Remark 4.6. The assumption that V be rational is important. For instance, consider the baby Verma module $M(\lambda)$ over $\mathfrak{gl}_2$ for $\lambda = 0$. This is a p-dimensional module with basis $v, fv, f^2v, \dots, f^{p-1}v$, where v is the highest weight vector and f is the matrix e_{21}. When t is specialized to the f-action, this gives an $\boldsymbol{\alpha}_p$-module isomorphic to J_p. However, if e is the matrix e_{12}, which is conjugate to f, then there are two highest weight vectors, as $e.v = 0$ and $e.(fv) = e.f.v = f.e.v - h.v = 0$. This means that when t is specialized to the e-action, we get an $\boldsymbol{\alpha}_p$-module isomorphic to $J_1 \oplus J_{p-1}$.

4.2. **Specializing to super vector spaces.** In this section, we will mainly consider objects in $\operatorname{Rep}\boldsymbol{\alpha}_p$ with indecomposable summands isomorphic to J_1, J_{p-1}, or J_p, so that the semisimplification is a super vector space. As such, we will remind the viewer to review section 3 for any notation.

For context to Theorem 4.7 , let $Y = \bigoplus_{i=1}^p n_i J_i$ be an object in $\operatorname{Rep}\boldsymbol{\alpha}_p$ where the j-th copy of J_i for $1 \le i \le p-1$ and $1 \le j \le n_i$ has basis

$$y_{j,1}^{(i)} \mapsto y_{j,2}^{(i)} \mapsto \cdots \mapsto y_{j,i}^{(i)}$$

Similarly, let $X = \bigoplus_{i=1}^p m_i J_i$, and $Z = \bigoplus_{i=1}^p k_i J_i$ be objects in $\operatorname{Rep}\boldsymbol{\alpha}_p$ with similarly labeled bases, using the letters 'x' and 'z' instead, respectively. Moreover, for X assume that only m_1, m_{p-1} and m_p can be nonzero. We can write their semisimplifications in isotypic decomposition as

$$\overline{X} = X_1 \otimes L_1 \oplus X_{p-1} \otimes L_{p-1}$$

$$\overline{Y} = \bigoplus_{i=1}^{p-1} Y_i \otimes L_i$$

$$\overline{Z} = \bigoplus_{i=1}^{p-1} Z_i \otimes L_i$$

where $X_1, X_{p-1}, X_p, Y_1, \ldots, Y_{p-1}, Z_1, \ldots, Z_{p-1}$ are vector spaces. Now, suppose that $\phi : X \otimes Y \to Z$ be a morphism in $\operatorname{Rep} \boldsymbol{\alpha}_p$. Because for each i with $1 \leq i \leq \frac{p-1}{2}$, the set $\{L_i, L_{p-i}\}$ is closed under tensoring with L_1 or L_{p-1}, semisimplicity implies that the semisimplification $\overline{\phi} : \overline{X} \otimes \overline{Y} \to \overline{Z}$ is determined by maps

$$\overline{\phi}_i : (X_1 \otimes L_1 \oplus X_{p-1} \otimes L_{p-1}) \otimes (Y_i \otimes L_i \oplus Y_{p-i} \otimes L_{p-i}) \to (Z_i \otimes L_i \oplus Z_{p-i} \otimes L_{p-i})$$

such that $\overline{\phi} = \bigoplus_{i=1}^{\frac{p-1}{2}} \overline{\phi}_i$. Finally, factoring out the L_i (that is, writing $Y_i \otimes L_i \oplus Y_{p-i} \otimes L_{p-i}$ as $(Y_i \oplus Y_{p-i} \otimes L_{p-1}) \otimes L_i$ and similarly for Z), we can interpret $\overline{\phi}_i$ as a map of super vector spaces from $(X_1 \oplus X_{p-1}) \otimes (Y_i \oplus Y_{p-i})$ to $Z_i \oplus Z_{p-i}$, where $X_{p-1}, Y_{p-i}, Z_{p-i}$ are in odd degree. With this interpretation, we have the following theorem:

Theorem 4.7. *There are bases* $\{\overline{x}_1^{(1)}, \overline{x}_2^{(1)}, \ldots, \overline{x}_{m_1}^{(1)}\}$ *of* X_1 *and* $\{\overline{x}_1^{(p-1)}, \overline{x}_2^{(p-1)}, \ldots, \overline{x}_{m_{p-1}}^{(p-1)}\}$ *of* X_{p-1} *and for each* i *there are bases*

(1) $\{\overline{y}_1^{(i)}, \overline{y}_2^{(i)}, \ldots, \overline{y}_{n_i}^{(i)}\}$ *of* Y_i *and* $\{\overline{y}_1^{(p-i)}, \overline{y}_2^{(p-i)}, \ldots, \overline{y}_{n_{p-i}}^{(p-i)}\}$ *of* Y_{p-i},
(2) $\{\overline{z}_1^{(i)}, \overline{z}_2^{(i)}, \ldots, \overline{z}_{k_i}^{(i)}\}$ *of* Z_i *and* $\{\overline{z}_1^{(p-i)}, \overline{z}_2^{(p-i)}, \ldots, \overline{z}_{k_{p-i}}^{(p-i)}\}$ *of* Z_{p-i},

such that $\overline{\phi}_i : (X_1 \oplus X_{p-1}) \otimes (Y_i \oplus Y_{p-i}) \to Z_i \oplus Z_{p-i}$ *is given by*

$$\overline{\phi}_i(\overline{x}_q^{(1)} \otimes \overline{y}_r^{(i)}) = \sum_{s=1}^{k_i} a_{qr}^s \overline{z}_s^{(i)};$$

$$\overline{\phi}_i(\overline{x}_q^{(1)} \otimes \overline{y}_r^{(p-i)}) = \sum_{s=1}^{k_{p-i}} b_{qr}^s \overline{z}_s^{(p-i)};$$

$$\overline{\phi}_i(\overline{x}_q^{(p-1)} \otimes \overline{y}_r^{(i)}) = \sum_{s=1}^{k_{p-i}} c_{qr}^s \overline{z}_s^{(p-i)};$$

$$\overline{\phi}_i(\overline{x}_q^{(p-1)} \otimes \overline{y}_r^{(p-i)}) = \sum_{s=1}^{k_i} d_{qr}^s \overline{z}_s^{(i)},$$

where we have four combinations listed, based on whether an even or odd vector is tensored with an even or odd vector, and where q, r *range over the appropriate ranges depending on whether which combination of even and odd was chosen. The structure constants above are given by:*

(1) a_{qr}^s *is the coefficient of* $z_{s,1}^{(i)}$ *in* $\phi(x_{q,1}^{(1)} \otimes y_{r,1}^{(i)})$;
(2) b_{qr}^s *is the coefficient of* $z_{s,1}^{(p-i)}$ *in* $\phi(x_{q,1}^{(1)} \otimes y_{r,1}^{(p-i)})$;
(3) c_{qr}^s *is the coefficient of* $z_{s,1}^{(p-i)}$ *in*

$$\phi\left(\sum_{l=1}^{i} \binom{i}{l} x_{q,l}^{(p-1)} \otimes y_{r,i+1-l}^{(i)}\right);$$

(4) d_{qr}^s *is the coefficient of* $z_{s,1}^{(i)}$ *in*

$$\phi\left(\sum_{l=1}^{p-i}\binom{p-i}{l}x_{q,l}^{(p-1)}\otimes y_{r,p-i+1-l}^{(p-i)}\right).$$

Proof. We can deduce the theorem by studying restrictions of ϕ to summands in $X\otimes Y$. For a fixed i, we have four possibilities: restricting ϕ to a J_1 from X and J_i from Y, to a J_1 from X and J_{p-i} from Y, to a J_{p-1} from X and J_i from Y, and to a J_{p-1} from X and J_{p-i} from Y (anything involving J_p will automatically vanish in the semisimplification).

For the first case, we have a map $\phi : J_1\otimes J_i\to Z$. Since the left hand side is isomorphic to J_i, the non-negligible piece comes from seeing where its generating vector maps to under ϕ. Therefore, we are interested decomposing this restriction into the sum of maps into the various the J_i terms in Z that are isomorphisms onto their image in Z plus negligible morphisms. Semisimplification will kill the negligible morphisms, so this gives the coefficients a_{qr}^s. A similar statement proves the second case, giving the coefficients b_{qr}^s.

The last two cases are a little trickier, but are a consequence of Theorem 2.2. In particular, when we restrict ϕ to a term of the form $J_{p-1}\otimes J_i$, we have $J_{p-1}\otimes J_i\cong J_{p-i}\oplus(k-1)J_p$. So the semisimplification of the map is determined by seeing where this J_{p-i} maps to in Z. Theorem 4.7 gives us a generating vector for this J_{p-i} in a suitable decomposition, so we just need to see which generating vectors in the given J_{p-i}'s in Z appear. Similarly, for the last case, this same argument works, just replacing i with $p-i$ and vice versa. □

We will often apply this theorem in two cases. The first is when $X=Y=Z$ is a Lie algebra in $\operatorname{Rep}\boldsymbol{\alpha}_p$ and ϕ is the bracket $B : X\otimes X\to X$. Applying the theorem in this instance tells us the structure constants of the Lie superalgebra $\overline{X}$. The other is when X is again a Lie algebra in $\operatorname{Rep}\boldsymbol{\alpha}_p$ and $Y=Z$ is a module over X and ϕ is the $\boldsymbol{\alpha}_p$-equivariant module map $\rho : X\otimes Y\to Y$. In this case, we get $\frac{p-1}{2}$-modules over the Lie superalgebra $\overline{X}$.

Remark 4.8. This theorem generalizes Proposition 3.4.1 in [Kan22].

Now, let's turn to general linear Lie algebras in $\operatorname{Rep}\boldsymbol{\alpha}_p$. Let V be an $m+n(p-1)$-dimensional vector space with the following ordered basis

$$\{v_1,v_2,\dots,v_m\}\cup\bigcup_{j=1}^{n}\{v_{j,1},v_{j,2},\dots v_{j,p-1}\},$$

where the ordering is given by reading the elements above from left to right in the obvious way as j increases from 1 to n. Use this basis to identify $\mathfrak{gl}(V)$ with $\mathfrak{gl}(m+n(p-1))$. Using the notation

$$\mathcal{J}_i := \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}_{i\times i},$$

define the matrix $x\in\mathfrak{gl}(m+n(p-1))$ by

$$x=\operatorname{diag}\big(\underbrace{\mathcal{J}_1,\dots,\mathcal{J}_1}_{m},\underbrace{\mathcal{J}_{p-1},\dots,\mathcal{J}_{p-1}}_{n}\big).$$

Then, (V,x) is an object in $\operatorname{Rep}\boldsymbol{\alpha}_p$ with decomposition $(V,x) = mJ_1 \oplus nJ_{p-1}$, with the i-th copy of J_1 spanned by v_i for $1 \leq i \leq m$, and the j-th copy of J_{p-1} is given by

$$v_{j,1} \mapsto v_{j,2} \mapsto \cdots \mapsto v_{j,p-1}$$

for $1 \leq j \leq n$. It follows that $(V,x)^*$ is an object in $\operatorname{Rep}\boldsymbol{\alpha}_p$ with decomposition into indecomposables as

$$\begin{aligned} J_1 &: v_1^*, v_2^*, \ldots, v_m^*; \\ J_{p-1} &: -v_{1,1}^* \hookleftarrow v_{1,2}^* \hookleftarrow \cdots \hookleftarrow -v_{1,p-2}^* \hookleftarrow v_{1,p-1}^* \\ &\quad\; -v_{2,1}^* \hookleftarrow v_{2,2}^* \hookleftarrow \cdots \hookleftarrow -v_{2,p-2}^* \hookleftarrow v_{1,p-1}^* \\ &\qquad\qquad\qquad \vdots \\ &\quad\; -v_{n,1}^* \hookleftarrow v_{n,2}^* \hookleftarrow \cdots \hookleftarrow -v_{n,p-2}^* \hookleftarrow v_{n,p-1}^*. \end{aligned}$$

Notice that the arrows are reversed for the dual vectors and the sign alternates. It will be useful to let the object X_i denote the i-th copy of J_1 and $X_{\overline{j}}$ to denote the j-th copy of J_{p-1} in our decomposition of V so that $V = \bigoplus_{i\in\mathbb{I}} X_i$, where $\mathbb{I} = \{1,\ldots,m,\overline{1},\ldots\overline{n}\}$. Putting this together, we can see that $(\mathfrak{gl}(m+n(p-1)), \operatorname{ad} x)$ is an object in $\operatorname{Rep}\boldsymbol{\alpha}_p$ as follows:

$$\left[\begin{array}{ccc|ccc} X_1\otimes X_1^* & \cdots & X_1\otimes X_m^* & X_1\otimes X_{\overline{1}}^* & \cdots & X_1\otimes X_{\overline{n}}^* \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_m\otimes X_1^* & \cdots & X_m\otimes X_m^* & X_m\otimes X_{\overline{1}}^* & \cdots & X_m\otimes X_{\overline{n}}^* \\ \hline X_{\overline{1}}\otimes X_1^* & \cdots & X_{\overline{1}}\otimes X_1^* & X_{\overline{1}}\otimes X_{\overline{1}}^* & \cdots & X_{\overline{1}}\otimes X_{\overline{n}}^* \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{\overline{n}}\otimes X_1^* & \cdots & X_{\overline{n}}\otimes X_m^* & X_{\overline{n}}\otimes X_{\overline{1}}^* & \cdots & X_{\overline{n}}\otimes X_{\overline{n}}^* \end{array}\right].$$

We can produce a partial basis for $(\mathfrak{gl}(m+n(p-1)), \operatorname{ad} x)$, ignoring any J_p terms that arise (which only happens in blocks in the lower right). Each $X_i \otimes X_j^*$ gives a J_1 spanned by $v_i \otimes v_j^*$. Each $X_i \otimes X_{\overline{j}}^*$ gives a J_{p-1} with basis

$$-v_i \otimes v_{j,1}^* \hookleftarrow v_i \otimes v_{j,2}^* \hookleftarrow \cdots \hookleftarrow -v_i \otimes v_{j,p-2}^* \hookleftarrow v_i \otimes v_{j,p-1}^*.$$

Similarly, each $X_{\overline{i}} \otimes X_j^*$ gives a J_{p-1} with basis

$$v_{i,1} \otimes v_j^* \mapsto v_{i,2} \otimes v_j^* \mapsto \cdots \mapsto v_{i,p-1} \otimes v_j^*.$$

Finally, each $X_{\overline{i}} \otimes X_{\overline{j}}^* = J_1 \oplus (p-2)J_p$. The J_p terms and any morphisms into them will vanish in the semisimplification, so we only care about the J_1. Theorem 2.2 tells us that a basis for this J_1 is given by

$$\sum_{k=1}^{p-1} v_{i,k} \otimes v_{j,k}^*.$$

Assembling all of this helps us prove the following theorem:

Theorem 4.9. *The semisimplification of* $(\mathfrak{gl}(m+n(p-1)), \operatorname{ad} x)$ *is* $\mathfrak{gl}(m|n)$.

Proof. We have already established the statement in much more generality. However, it will be useful to explicitly prove it using computations using Theorem 4.7. The key idea that we wish to stress is that each J_{p-1} merges under the semisimplification functor to give a one-dimensional odd subspace.

First, let's compute what the bracket looks like in $\boldsymbol{\alpha}_p$-equivariant fashion, before semisimplification and ignoring any J_p terms. We have a few cases to consider:

(1) the first case is if we are bracketing a J_1 from the top left with something. We have:

$$B(v_i \otimes v_j^* \otimes v_k \otimes v_l^*) = \delta_{jk} v_i \otimes v_l^* - \delta_{li} v_k \otimes v_l^*;$$
$$B((v_i \otimes v_j^*) \otimes (-v_k \otimes v_{l,1}^* \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid v_k \otimes v_{l,p-1})) = -\delta_{jk} v_i \otimes v_{l,1}^* \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid \delta_{jk} v_i \otimes v_{l,p-1};$$
$$B((v_i \otimes v_j^*) \otimes (v_{k,1} \otimes v_l^* \mapsto \cdots \mapsto v_{k,p-1} \otimes v_l^*)) = -\delta_{li} v_{k,1} \otimes v_j^* \mapsto \cdots \mapsto -\delta_{li} v_{k,p-1} \otimes v_j^*;$$
$$B\left((v_i \otimes v_j^*) \otimes \sum_{s=1}^{p-1} v_{k,s} \otimes v_{l,s}^*\right) = 0.$$

The first line corresponds to bracketing with another J_1 from the top left. The middle two lines are from bracketing with a J_{p-1} from the top right or bottom left, respectively. The last line is from bracketing with a J_1 from the bottom right.

(2) the second case is if we are bracketing a J_{p-1} from the top right with something. By skew-symmetry of the bracket and the fact that bracketing two things in the top right will give zero (triangular decomposition property), we only need to consider pairing with something from the bottom left or bottom right. When we pair with the bottom left, we are looking at a bracket on $J_{p-1} \otimes J_{p-1}$. The first J_{p-1} here is given by

$$J_{p-1} = -v_i \otimes v_{j,1}^* \leftarrow\!\shortmid v_i \otimes v_{j,2}^* \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid -v_i \otimes v_{j,p-2}^* \leftarrow\!\shortmid v_i \otimes v_{j,p-1}^*$$

and the second J_{p-1} is given by

$$J_{p-1} = v_{k,1} \otimes v_l^* \mapsto v_{k,2} \otimes v_l^* \mapsto \cdots \mapsto v_{k,p-1} \otimes v_l^*.$$

By Theorem 2.2, there is a J_1 in here spanned by

$$w \coloneqq \sum_{s=1}^{p-1} (-1)^s ((-1)^{s-1} v_i \otimes v_{j,p-s}^*) \otimes (v_{k,p-s} \otimes v_l^*).$$

Applying the bracket gives us:

$$\begin{aligned} B(w) &= -(p-1)\delta_{jk} v_i \otimes v_l^* + \delta_{li} \left(\sum_{s=1}^{p-1} v_{k,p-s} \otimes v_{j,p-s}^*\right) \\ &= \delta_{jk} v_i \otimes v_l^* + \delta_{li} \left(\sum_{s=1}^{p-1} v_{k,s} \otimes v_{j,s}^*\right). \end{aligned}$$

Notice the miraculous $-(p-1) = 1$ factor, which makes this resemble like something of the form $[x, y] = xy + yx$ for x, y both odd, especially under the perspective that after semisimplification a J_{p-1} merges to form an odd subspace.

The other case is when we pair a J_{p-1} with a J_1 from the bottom left. We have:

$$B\left((-v_i \otimes v_{j,1}^* \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid v_i \otimes v_{j,p-1}^*) \otimes \left(\sum_{s=1}^{p-1} v_{k,s} \otimes v_{l,s}^*\right)\right)$$
$$= -\delta_{jk} v_i \otimes v_{l,1}^* \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid \delta_{jk} v_i \otimes v_{l,p-1}^*.$$

(3) The third case is if we are bracketing a J_{p-1} from the bottom left with something. Again, by skew-symmetry and the triangular decomposition, we just need to bracket with a J_1 from the bottom right. We have

$$B\left((v_{i,1}\otimes v_j^* \mapsto \cdots \mapsto v_{i,p-1}\otimes v_j^*)\otimes\left(\sum_{s=1}^{p-1} v_{k,s}\otimes v_{l,s}^*\right)\right)$$
$$= -\delta_{li}(v_{k,1}\otimes v_j^* \mapsto \cdots \mapsto v_{k,p-1}\otimes v_j^*)$$

(4) The final case is bracket a J_1 from the bottom left with something. We just need to check bracketing with another such J_1. We get:

$$B\left(\left(\sum_{s=1}^{p-1} v_{i,s}\otimes v_{j,s}^*\right)\otimes\left(\sum_{s=1}^{p-1} v_{k,s}\otimes v_{l,s}^*\right)\right)$$
$$= \delta_{jk}\left(\sum_{s=1}^{p-1} v_{i,s}\otimes v_{l,s}^*\right) - \delta_{li}\left(\sum_{s=1}^{p-1} v_{k,s}\otimes v_{j,s}^*\right)$$

By now, we see these commutation relations clearly resemble those of $\mathfrak{gl}(m|n)$. And indeed, if we apply Theorem 4.7, we get that the basis vector afforded by $X_i\otimes X_j^*$ is just e_{ij} for i,j and the bracket is given by $[e_{ij},e_{kl}] = \delta_{jk}e_{il} - (-1)^{(|i|+|j|)(|k|+|l|)}\delta_{li}e_{kj}$, recalling the parity function from section 3.1. This proves the theorem. □

The utility of this theorem is that we can now easily study a map involving $(\mathfrak{gl}(m+n(p-1)), \operatorname{ad} x)$ before and after semisimplification. Our first application will be for understanding what happens when we semisimplify the Chevalley automorphism.

Proposition 4.10. *The Chevalley automorphism $\theta : \mathfrak{gl}(m+n(p-1)) \to \mathfrak{gl}(m+n(p-1))$, given by $\theta(x) = -x^T$ is a Lie algebra isomorphism $(\mathfrak{gl}(m+n(p-1)), \operatorname{ad} x) \to (\mathfrak{gl}(m+n(p-1)), \operatorname{ad}\theta(x))$ in* $\operatorname{Rep}\boldsymbol{\alpha}_p$.

Proof. It suffices to show that this map is $\boldsymbol{\alpha}_p$-equivariant. We have

$$\theta(t.y) = \theta([x,y]) = [\theta(x),\theta(y)] = t.\theta(y).$$

□

Notice that this proof actually holds for any Lie algebra automorphism ψ, not just the Chevalley automorphism.

Theorem 4.11. *The semisimplification of the Chevalley automorphism $\theta : (\mathfrak{gl}(m+n(p-1)), \operatorname{ad} x) \to (\mathfrak{gl}(m+n(p-1)), \operatorname{ad}\theta(x))$ is the Chevalley automorphism $\theta : \mathfrak{gl}(m|n) \to \mathfrak{gl}(m|n)$.*

Proof. At the level of basis vectors, we have

$$\begin{aligned} \theta(v_i\otimes v_j^*) &= -v_j\otimes v_i^*;\\ \theta(-v_i\otimes v_{j,1}^* \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid v_i\otimes v_{j,p-1}^*) &= v_{j,1}\otimes v_i^* \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid -v_{j,p-1}\otimes v_i^*\\ \theta(v_{i,1}\otimes v_j^* \mapsto \cdots \mapsto v_{i,p-1}\otimes v_j^*) &= -v_j\otimes v_{i,1}^* \mapsto \cdots \mapsto -v_j\otimes v_{i,p-1}^*\\ \theta\left(\sum_{s=1}^{p-1}(-1)^s v_{i,s}\otimes v_{j,s}^*\right) &= -\sum_{s=1}^{p-1}(-1)^s v_{j,s}\otimes v_{i,s}^*. \end{aligned} \tag{4.1}$$

Now, the object $(\mathfrak{gl}(m+n(p-1)), \operatorname{ad}\theta(x))$ has a decomposition into decomposables where the J_1's are spanned by $v_i \otimes v_j^*$ (top-left) and $\sum_{s=1}^{p-1}(-1)^s v_{i,s} \otimes v_{j,s}^*$ (bottom right), and where the J_{p-1}'s are given by

$$v_{i,1} \otimes v_j^* \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid -v_{i,p-1} \otimes v_j^*$$

in the bottom left and by

$$v_i \otimes v_{j,1}^* \mapsto \cdots \mapsto v_i \otimes v_{j,p-1}^*$$

in the top right. In a calculation similar to the one found in the proof of Theorem 4.9, the basis of the semisimplification afforded by Theorem 4.7 when applied to this decomposition and the bracket is also $\{e_{ij}\}_{i,j\in\mathbb{I}}$ in the obvious way. Therefore, by looking at (4.1) line by line we get:

$$\overline{\theta}(e_{ij}) = \begin{cases} -e_{ji} & |i| = |j| = 0 \\ e_{ji} & |i| = 0, |j| = 1 \\ -e_{ji} & |i| = 1, |j| = 0 \\ -e_{ji} & |i| = |j| = 1 \end{cases}$$

for $i, j \in \mathbb{I}$. The map $\overline{\theta}$ is then precisely the Chevalley automorphism $\theta(x) = -x^{ST}$ on $\mathfrak{gl}(m|n)$. This proves the theorem. □

Proposition 4.12. *Let $L(\lambda)$ be a simple $\mathfrak{gl}(m+n(p-1))$-module with representation $\rho : \mathfrak{gl}(m+n(p-1)) \to \mathfrak{gl}(L(\lambda))$ so that $(L(\lambda), \rho(x))$ is a module over $(\mathfrak{gl}(m+n(p-1)), \operatorname{ad} x)$, and $(L(\lambda), \rho(\theta(x)))$ is a module over $(\mathfrak{gl}(m+n(p-1)), \operatorname{ad}\theta(x))$. Then, the Shapovalov form $S : L(\lambda) \otimes L(\lambda) \to \mathbb{1}$ is a $\boldsymbol{\alpha}_p$-equivariant map $S : (L(\lambda), \rho(x)) \otimes L(\lambda), \rho(\theta(x))) \to \mathbb{1}$, and the semisimplification of $\overline{S}$ is a non-degenerate symmetric contravariant form.*

Proof. It is clear by contravariance that $S(t.(v \otimes w)) = S(\rho(x)v \otimes w + v \otimes \rho(\theta(x))w) = 0 = t.S(v \otimes w)$, so this proves the first statement.

For the second statement, the contravariance condition can be written categorically by saying there is a module map $W \otimes W^\theta \to \mathbb{1}$, where W^θ is the module given by first twisting $\mathfrak{gl}(m+n(p-1))$ by the automorphism θ and then acting. By Theorem 4.11 and the symmetric monoidal property of the semisimplification functor, it follows that $\overline{S}$ is contravariant as well. The non-degeneracy and symmetry are also obviously preserved. □

Corollary 4.13. *The semisimplification of $(L(\lambda), \rho(x))$ is a semisimple $\mathfrak{gl}(m|n)$-module.*

Proof. The semisimplification has a non-degenerate symmetric convariant form (the semisimplification of the Shapovalov form). Therefore, by Proposition 3.1, it is semisimple. □

We expect a totally analogous statement to hold for any nilpotent x with $x^p = 0$ and replacing $\mathfrak{gl}(m|n)$ with the Lie algebra $\mathfrak{gl}(\overline{V})$ in the Verlinde category, and it should be straightforward to prove. We imagine it would boil down to coming up with a categorical definition of the transpose operator. Perhaps appealing to the theory of contragredient Lie algebra in symmetric tensor categories, as in [APS24], might be another starting point. Then, one can just follow the steps in the proof verbatim (maybe some care is required with orthogonal complements. Much of the discussion in section 3.2.2 is also relevant because there is a Steinberg tensor product theorem in the Verlinde category for $GL(\overline{V})$ modules (see [Kan25]).

We use this approach because in this paper we are interested in examples of calculations using the semisimplification functor. Such calculations are much difficult with the Verlinde category because the objects are no longer vector spaces.

Finally, we conclude this section with a statement about semisimplification of modules. Let $(L(\lambda), \rho(x))$ be a module over $(\mathfrak{gl}(m+n(p-1)), \operatorname{ad} x)$ in $\operatorname{Rep} \boldsymbol{\alpha}_p$, where $\rho : \mathfrak{gl}(m+n(p-1)) \to \mathfrak{gl}(L(\lambda))$ is the associated representation. Write

$$\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_m\epsilon_m + \left(\sum_{s=1}^{p-1} \lambda_{1,s}\epsilon_{1,s}\right) + \cdots + \left(\sum_{s=1}^{p-1} \lambda_{n,s}\epsilon_{n,s}\right),$$

where $\epsilon_{j,s}$ is the dual vector in $\mathfrak{h}^*$ to $v_{j,s} \otimes v_{j,s}^*$. Finally, define $\overline{\lambda}$ in $\mathfrak{h}^*$ of $\mathfrak{gl}(m|n)$ by

$$\overline{\lambda} \coloneqq \left(\lambda_1, \ldots, \lambda_m, \sum_{s=1}^{p-1} \lambda_{1,s}, \ldots, \sum_{s=1}^{p-1} \lambda_{n,s}\right)$$

Theorem 4.14. *Suppose the highest weight v of $L(\lambda)$ generates a J_r in some decomposition of $L(\lambda)$ into indecomposables under the $\rho(x)$ action. If $r \neq p$, then the semisimplification of $(L(\lambda), \rho(x))$ contains the module $L(\overline{\lambda}) \otimes L_r$ as a direct summand.*

Proof. This is a consequence of Theorem 4.7, Theorem 4.9, and Corollary 4.13. In particular, the vector $\sum_{s=1}^{p-1} v_{i,s} \otimes v_{i,s}^*$ acts as a scalar on v and therefore J_r. This scalar is $\sum_{s=1}^{p-1} \lambda_{i,s}$. It follows that the vector it yields, which is $e_{\overline{ii}}$ in $\mathfrak{gl}(m|n)$, acts as the same scalar. It is clear then that there is a highest weight vector with weight $\overline{\lambda}$, so it must generate a simple module by Corollary 4.13. □

More generally, our goal is to search for vectors in $L(\lambda)$ such that after semisimplification they yield highest weight vectors. By Corollary 4.13, the module generated by this highest weight vector will be simple.

5. Explicit computations using semisimplification

In this section, we conclude by explicitly computing semisimplifications in low rank and low characteristic. We use the Weyl modules program by Stephen Doty (for more detail see [Dot24]). The decomposition algorithm for modules is described in Section 5.1 along with detailed pseudocode and correctness proofs.

5.1. **Decomposition algorithm.** Let $\mathbb{K}$ be a field, M be a finite-dimensional module, and $t : M \to M$ a fixed nilpotent operator. Let $\mathcal{B} = (b_1, \ldots, b_N)$ be a basis of M, totally ordered by weight (lowest to highest if t is a raising operator; reverse if t is lowering). A *Jordan chain* (or block) of length i is a sequence

$$J_i(v) : \quad v,\, tv,\, \ldots,\, t^{i-1}v \quad \text{with } t^i v = 0,\ t^{i-1}v \neq 0.$$

We maintain a set of chains $\mathcal{J} = \{J^{(1)}, \ldots, J^{(r)}\}$ and a multiset $\mathcal{B}_{\mathcal{J}}$ of the vectors in the set $\mathcal{J}$:

$$\mathcal{B}_{\mathcal{J}} \coloneqq \bigsqcup_{J\in\mathcal{J}} \bigsqcup_{u\in J} \{u\} \subseteq M,$$

i.e. $\mathcal{B}_{\mathcal{J}}$ contains every vector from every chain in $\mathcal{J}$. The algorithm iterates over the ordered basis B. Whenever the current seed $v = b_i$ already lies in $\operatorname{span}(\mathcal{B}_{\mathcal{J}})$ we skip it. Otherwise, we attempt to extend v to a chain by repeatedly applying t until either we hit 0 (success, in which case this new chain in included in $\mathcal{J}$) or we encounter the *first dependency* against

$\mathrm{span}(\mathcal{B}_{\mathcal{J}})$. Let this dependency be expressed as

$$t^{\ell} v \;=\; \sum_{j=1}^{m} e_j \, t^{\alpha_j} a_j \qquad \bigl(t^{\alpha_j} a_j \in \mathcal{B}_{\mathcal{J}}\bigr).$$

The algorithm records the precise form of this dependence in a container called *Relation*. This container stores information on the first dependency encountered, the dependent vector, the coefficients, and the specific earlier chain nodes it depends on.

We then *resolve* this first dependency by one of two cases, determined by the minimum index $\delta_{\min} := \min_j \alpha_j$.

- **Case 1 ($\delta_{\min} > \ell$):** All right-hand-side nodes lie strictly *ahead* of $t^{\ell} v$. We reseed with $w := v - \sum_j e_j \, t^{\alpha_j - \ell} a_j$, whose chain terminates at length ℓ and whose earlier nodes remain independent of $\mathcal{B}_{\mathcal{J}}$. We ignore the partial chain from v and add the chain from w.
- **Case 2 ($\delta_{\min} \leq \ell$):** Some right-hand-side node is not ahead of $t^{\ell} v$. We *exchange* one earlier shortest chain for the greedy chain from v, then include the chain from the auxiliary seed $w := t^{\ell - \delta_{\min}} v - \sum_j e_j \, t^{\alpha_j - \delta_{\min}} a_j$, which produces an additional chain of length $\delta_{\min}$ to include in $\mathcal{J}$.

Proceeding through all b_i in order yields a set $\mathcal{J}$ of Jordan chains which represent a decomposition of M whose concatenated nodes $\mathcal{B}_{\mathcal{J}}$ form a basis of M.

Remark 5.1 (Importance of ordering B). The correctness of the algorithm requires that the basis $\mathcal{B}$ is totally ordered by weight. Let us consider the case when t is a raising operator, and the basis is ordered starting from the lowest weight vector. If the basis elements are not processed in increasing order, then consider the first vector in the ordered basis, b_{lw} that is not processed. Since subsequent vectors are higher weight, b_{lw} will never be included in subsequent Jordan chains. Therefore, processing basis elements in increasing order guarantees that when the algorithm terminates, $\mathcal{B}_{\mathcal{J}}$ is a valid basis of M.

We now describe the routines *GreedyChain*, *Resolve*, and the main loop *JordanDecompose*, along with the correctness proof.

Algorithm 1 GREEDYCHAIN(v)

Require: Seed $v \in M$, current concatenated set $\mathcal{B}_{\mathcal{J}}$.
Ensure: A chain $S = [v, tv, \ldots]$ until zero or first dependency. If a first dependency occurs at step i, also return its linear relation.

$S \leftarrow [\,]$; $\quad u \leftarrow v$
loop
  Append u to S
  if $tu = 0$ **then**
    return (S, `NoRelation`)
  else if $tu \in \mathrm{span}(\mathcal{B}_{\mathcal{J}})$ **then** ▷ first dependency detected
    Find coefficients e_j and nodes $t^{\alpha_j} a_j \in \mathcal{B}_{\mathcal{J}}$ s.t. $tu = \sum_{j=1}^{m} e_j \, t^{\alpha_j} a_j$.
    return (S, `Relation`($\{(e_j, \alpha_j, a_j)\}_{j=1}^{m}$))
  else
    $u \leftarrow tu$
  end if
end loop

Algorithm 2 RESOLVE(S, `Relation`)

Require: Chain $S = [v, tv, \ldots, t^{i-1}v]$; first dependency $t^i v = \sum_{j=1}^{m} e_j\, t^{\alpha_j} a_j$, with each $t^{\alpha_j} a_j$ a node inside a prior chain $J^{(j)}_{k_j} = (a_j, ta_j, \ldots, t^{k_j - 1} a_j)$.

Let $\delta_j \leftarrow \alpha_j$ and $\delta_{\min} \leftarrow \min_j \delta_j$.
if $\delta_{\min} > i$ **then** ▷ Case 1: earlier nodes are strictly behind the RHS nodes
  Set $o_j \leftarrow \delta_j - i > 0$ for all j.
  Define $w \leftarrow v - \sum_{j=1}^{m} e_j\, t^{o_j} a_j$.
  return `Case1`(w) ▷ Discard S; add chain from w
else ▷ Case 2: some RHS node not ahead of $t^i v$
  Let j^* be an index with $\delta_{j^*} = \delta_{\min}$.
  Set $o \leftarrow i - \delta_{\min} \geq 0$, and $o_j \leftarrow \delta_j - \delta_{\min} \geq 0$.
  Define $w \leftarrow t^o v - \sum_{j=1}^{m} e_j\, t^{o_j} a_j$.
  return `Case2`(j^*, S, w) ▷ Replace prior $J^{(j^*)}$ with S, then add chain from w
end if

Algorithm 3 JORDANDECOMPOSE($\mathcal{B}, t$)

Require: Ordered basis $\mathcal{B} = (b_1, \ldots, b_N)$, nilpotent t.
Ensure: A set of Jordan chains $\mathcal{J}$ whose concatenated vectors $\mathcal{B}_{\mathcal{J}}$ form a basis of M.

$\mathcal{J} \leftarrow \emptyset$, $\mathcal{B}_{\mathcal{J}} \leftarrow \emptyset$
for $i = 1$ **to** N **do** ▷ iterate in weight order (reverse for lowering operators)
  $v \leftarrow b_i$
  if $v \in \operatorname{span}(\mathcal{B}_{\mathcal{J}})$ **then**
    continue
  end if
  $(S, \texttt{Rel}) \leftarrow$ GREEDYCHAIN(v)
  if `Rel` = `NoRelation` **then**
    Add chain S to $\mathcal{J}$; update $\mathcal{B}_{\mathcal{J}} \leftarrow \mathcal{B}_{\mathcal{J}} \sqcup S$
  else
    `Outcome` $\leftarrow$ RESOLVE(S, `Rel`)
    if `Outcome` = `Case1`(w) **then**
      $(S_{\text{new}}, \_) \leftarrow$ GREEDYCHAIN(w)
      Add S_{new} to $\mathcal{J}$; update $\mathcal{B}_{\mathcal{J}} \leftarrow \mathcal{B}_{\mathcal{J}} \sqcup S_{\text{new}}$
    else ▷ `Case2`(j^*, S, w)
      Replace the prior chain $J^{(j^*)} \in \mathcal{J}$ by S; update $\mathcal{B}_{\mathcal{J}}$ accordingly
      $(S_{\text{new}}, \_) \leftarrow$ GREEDYCHAIN(w)
      Add S_{new} to $\mathcal{J}$; update $\mathcal{B}_{\mathcal{J}} \leftarrow \mathcal{B}_{\mathcal{J}} \sqcup S_{\text{new}}$
    end if
  end if
end for
return $\mathcal{J}$

In order to prove the correctness of Algorithm 3 we establish the following lemmas.

Lemma 5.2 (First dependency implies earlier independence)**.** *If $t^r v \notin \operatorname{span}(\mathcal{B}_{\mathcal{J}})$ for $0 \leq r \leq \ell - 1$ and $t^\ell v \in \operatorname{span}(\mathcal{B}_{\mathcal{J}})$, then the chain $(v, tv, \ldots, t^{\ell-1}v)$ is independent of $\mathcal{B}_{\mathcal{J}}$.*

Proof. If a nontrivial linear dependence existed using some $t^r v$ with $r < \ell$, we would contradict minimality of ℓ as the first dependent index. □

Lemma 5.3 (Extension for Case 1)**.** *With $\delta_{\min} > \ell$, the seed $w = v - \sum_j e_j t^{\delta_j - \ell} a_j$ satisfies $t^\ell w = 0$, and its earlier nodes avoid* $\operatorname{span}(\mathcal{B}_{\mathcal{J}})$*. Moreover, $v \in \operatorname{span}(\mathcal{B}_{\mathcal{J}_{\text{new}}})$.*

Proof. Since the relation at index ℓ is exactly $t^\ell v = \sum_j e_j t^{\delta_j} a_j$, we have by construction,

$$t^\ell w \;=\; t^\ell v - \sum_{j=1}^{m} e_j\, t^{\delta_j} a_j \;=\; 0,$$

for $0 \le r < \ell$,

$$t^r w = t^r v - \sum_{j=1}^{m} e_j\, t^{\delta_j - \ell + r} a_j.$$

Since $\delta_j \ge \delta_{\min} > \ell$, we have $1 \le \delta_j - \ell + r < \delta_j$. Therefore, each correction term $t^{\delta_j - \ell + r} a_j \in J^{(j)} \in \mathcal{B}_{\mathcal{J}}$. Because ℓ is minimal, every $t^r v$ with $r < \ell$ is nonzero and independent of $\mathcal{B}_{\mathcal{J}}$. Subtracting additional correction terms from $\mathcal{B}_{\mathcal{J}}$ cannot create a new dependency at these indices. Thus the chain $(w, tw, \ldots, t^{\ell-1} w)$ is nonzero and independent of $\mathcal{B}_{\mathcal{J}}$. Finally,

$$v \;=\; w + \sum_{j=1}^{m} e_j\, t^{\delta_j - \ell} a_j,$$

where the summation lies in $\operatorname{span}(\mathcal{B}_{\mathcal{J}})$. Hence $v \in \operatorname{span}(\mathcal{B}_{\mathcal{J}_{\text{new}}} = \mathcal{B}_{\mathcal{J}} \sqcup S_{\text{new}})$, as required. □

Lemma 5.4 (Exchange + extension for Case 2)**.** *With $\delta_{\min} \le \ell$, replacing the earlier chain $J^{(j^*)}$ by the greedy chain S and then adding the chain from w preserves independence and increases coverage of $\mathcal{B}_{\mathcal{J}_{new}}$ to include v.*

Proof. By Lemma 5.2, the greedy chain $S = (v, tv, \ldots, t^{\ell-1} v)$ is independent of $\mathcal{B}_{\mathcal{J}}$ up to its first dependency. Now consider

$$w = t^{o} v - \sum_{j=1}^{m} e_j\, t^{o_j} a_j, \quad o = \ell - \delta_{\min},\; o_j = \alpha_j - \delta_{\min}.$$

Then

$$t^r w = t^r t^o v - \sum_j e_j t^{o_j + r} a_j = t^{\ell + r - \delta_{\min}} v - \sum_j e_j t^{\alpha_j + r - \delta_{\min}} a_j,$$

For $r = \delta_{\min}$, we have $t^r w = 0$. So the chain from w terminates at height $\delta_{\min}$.

For $0 \le r < \delta_{\min}$ we compute

$$o_j + r \;=\; (\alpha_j - \delta_{\min}) + r \;\le\; \alpha_j - 1 \;<\; \alpha_j.$$

Thus each correction term $t^{o_j + r} a_j$ lies strictly before the node $t^{\alpha_j} a_j$ from the relation in its respective chain $J^{(j)}$. Furthermore, $\ell + r - \delta_{\min} < \ell$. Therefore, the minimality of ℓ guarantees that $t^{\ell + r - \delta_{\min}} v$ is non zero and the newly generated chain from w is independent of the previously established vectors in $\mathcal{B}_{\mathcal{J}}$. This also implies linear independence from $\mathcal{B}_{\mathcal{J}} \setminus J^{(j^*)}$. Therefore, the replacement step

$$\mathcal{B}_{\mathcal{J}}' \;:=\; (\mathcal{B}_{\mathcal{J}} \setminus J^{(j^*)}) \;\sqcup\; \{v, tv, \ldots, t^{\ell-1} v\}$$

preserves linear independence and also spans v. Now, consider the chain

$$S_{\text{new}} \;:=\; \{w, tw, \ldots, t^{\delta_{\min} - 1} w\}.$$

If there was a nontrivial linear dependence of $t^r w$ $(1 \le r < \delta_{\min})$ and the vectors in $\mathcal{B}_{\mathcal{J}}'$, then from the definition of w, this would force a dependence in $\mathcal{B}_{\mathcal{J}}'$. This contradicts the independence of $\mathcal{B}_{\mathcal{J}}'$. Therefore, S_{new} is independent of $\mathcal{B}_{\mathcal{J}}'$, implying that

$$\mathcal{B}_{\mathcal{J}_{\text{new}}} = \mathcal{B}_{\mathcal{J}}' \sqcup S_{\text{new}}$$

preserves linear independence and also spans v. Furthermore, consider the terms in J^{j^*} that have been removed from $\mathcal{B}_{\mathcal{J}}$. We have for $i < \delta_{\text{new}}$

$$t^i w = t^i t^o v - \sum_j e_j t^{o_j + i} a_j.$$

Since for the minimum index j^*, $o_{j^*} + i = \alpha_{j^*} - \delta_{\min} + i = i$, we have $t^i a_{j^*}$ is linearly dependent on $t^i w$, $t^{i+o} v$ and terms from $J^{(j \neq j^*)}$. Therefore, the terms from J^{j^*} are still spanned by $\mathcal{B}_{\mathcal{J}_{\text{new}}}$. Therefore, the removal and extension operation on $\mathcal{B}_{\mathcal{J}}$ does not affect the vectors spanned originally by $\mathcal{B}_{\mathcal{J}}$. This concludes the proof that vectors covered by $\mathcal{B}_{\mathcal{J}}$ are still covered by $\mathcal{B}_{\mathcal{J}_{\text{new}}}$. Furthermore, $\mathcal{B}_{\mathcal{J}_{\text{new}}}$ is independent and spans v. □

We now prove the correctness for Algorithm 3 using three invariants:

(I1) **Independence:** $\mathcal{B}_{\mathcal{J}}$ is linearly independent at every loop head,
(I2) **Coverage of processed seeds:** For all $j < i$, $b_j \in \operatorname{span}(\mathcal{B}_{\mathcal{J}})$,
(I3) **Chain-closure:** Each $J \in \mathcal{J}$ is a valid Jordan block terminating at zero.

Initialization: Before the first iteration, $\mathcal{J} = \emptyset$, $\mathcal{B}_{\mathcal{J}} = \emptyset$: (I1)–(I3) hold trivially.

Maintenance: Assume (I1)–(I3) hold at the start of iteration i. If $b_i \in \operatorname{span}(\mathcal{B}_{\mathcal{J}})$ then no changes are made. (I1)–(I3) remain true, and (I2) extends to $j = i$. We now consider the case when $b_i \notin \operatorname{span}(\mathcal{B}_{\mathcal{J}})$. Algorithm 1 gives us two subcases.

If we encounter a terminal zero before any dependency, then Then $S = (b_i, \dots, t^{\ell-1} b_i)$ has all nodes outside $\operatorname{span}(\mathcal{B}_{\mathcal{J}})$, hence adding S preserves (I1). By construction $t^\ell b_i = 0$, so (I3) holds. Since $b_i \in \operatorname{span}(S) \subseteq \operatorname{span}(\mathcal{B}_{\mathcal{J}} \cup S)$, (I2) holds for index i.

If we encounter the first dependency at $t^\ell b_i = \sum_j e_j t^{\alpha_j} a_j$, then by minimality of ℓ, the nodes $b_i, tb_i, \dots, t^{\ell-1} b_i$ are independent from $\mathcal{B}_{\mathcal{J}}$. Let $\delta_j = \alpha_j$ and $\delta_{\min} = \min_j \delta_j$. If $(\delta_{\min} > \ell)$ then from Lemma 5.3 the chain from w is independent of $\mathcal{B}_{\mathcal{J}}$, preserving (I1). Furthermore, each chain terminates at zero, so (I3) holds. Also, $b_i \in \operatorname{span}(\mathcal{B}_{\mathcal{J}} \cup \{w, tw, \dots\}, t^{\ell-1} w)$, resulting in (I2) holding. For the case when $(\delta_{\min} \le \ell$, from Lemma 5.4, (I1) holds. Since all chains still terminate at zero, (I3) holds. Furthermore b_i lies in the updated span, so (I2) holds.

Hence (I1)–(I3) are maintained.

Termination: The loop runs for N iterations and terminates. At exit, (I2) implies $\operatorname{span}(\mathcal{B}_{\mathcal{J}}) = M$ and combined with (I1) we have that $\mathcal{B}_{\mathcal{J}}$ is a basis of M. Together with (I3), this proves that $\mathcal{J}$ is a valid Jordan decomposition under the action of t.

Time complexity. Let $N = \dim M$, and let L be the maximal chain length (in characteristic p, $L \le p$). Each seed advances at most L steps before zero or first dependency. With standard GAP linear algebra routines, membership tests/solves are $O(N^3)$, yielding a worst case complexity of $O(LN^4)$.

5.2. **Semisimplifications of $\mathfrak{gl}(3)$-modules in characteristic 3.** In this section we compute the semisimplification of simple $\mathfrak{sl}(3)$ modules, $L(\lambda)$ giving $\mathfrak{sl}(1|1)$ modules which can then be lifted to $\mathfrak{gl}(1|1)$ modules. The simple $\mathfrak{sl}(3)$ modules are given by the $3^2 = 9$ restricted dominant integral weights. We first use the algorithm provided in Section 5.1 to decompose each $L(\lambda)$ into Jordan blocks (w.r.t e_{32}) before semisimplification. The relevant decompositions and actions maps are shown in the Appendix (see A.1).

Case	λ	Decomposition
1	$(2,2,2)$	J_1
2	$(2,2,1)$	$J_1 \oplus J_2$
3	$(2,2,0)$	$J_1 \oplus J_2 \oplus J_3$
4	$(2,1,1)$	$J_1 \oplus J_2$
5	$(2,1,0)$	$2J_2 \oplus J_3$
6	$(2,1,-1)$	$J_1 \oplus J_2 \oplus 4J_3$
7	$(2,0,0)$	$J_1 \oplus J_2 \oplus J_3$
8	$(2,0,-1)$	$J_1 \oplus J_2 \oplus 4J_3$
9	$(2,0,-2)$	$9J_3$

TABLE 1. Decomposition of $L(\lambda)$ for $n = 3, p = 3$

1. $\lambda = (2,2,2)$
   In this case we see that the semisimplification is obviously simple as it is $(1|0)$ dimensional. Therefore, $\overline{L(\lambda)} = L(\overline{\lambda})$.
2. $\lambda = (2,2,1)$
   We can examine the action maps in the appendix to verify that the semisimplification is simple. Therefore, $\overline{L(\lambda)} = L(\overline{\lambda})$.
3. $\lambda = (2,2,0)$
   We can examine the action maps in the appendix to verify that the semisimplification is simple. Therefore, $\overline{L(\lambda)} = L(\overline{\lambda})$.
4. $\lambda = (2,1,1)$
   We can examine the action maps in the appendix to verify that the semisimplification is simple. Therefore, $\overline{L(\lambda)} = L(\overline{\lambda})$.
5. $\lambda = (2,1,0)$
   We can examine the action maps and note that there are two submodules. The two submodules have highest weight $\lambda_1 = (2,1,0)$ and $\lambda_2 = (0,2,1)$. Therefore, $\overline{L(\lambda)} = L(\overline{\lambda_1}) \oplus L(\overline{\lambda_2})$.
6. $\lambda = (2,1,-1)$
   We can examine the action maps in the appendix to verify that the semisimplification is simple. Therefore, $\overline{L(\lambda)} = L(\overline{\lambda})$.
7. $\lambda = (2,0,0)$
   We can examine the action maps in the appendix to verify that the semisimplification is simple. Therefore, $\overline{L(\lambda)} = L(\overline{\lambda})$.
8. $\lambda = (2,0,-1)$
   We can examine the action maps in the appendix to verify that the semisimplification is simple. Therefore, $\overline{L(\lambda)} = L(\overline{\lambda})$.
9. $\lambda = (2,0,-2)$
   In this case the decomposition consists of only J_3 meaning the semisimplification is the zero-map.

5.3. **Semisimplifications of $\mathfrak{gl}(4)$-modules in characteristic 3.** We use a similar approach of computing the semisimplification of simple $\mathfrak{sl}(4)$ modules giving $\mathfrak{sl}(2|1)$ modules. Additionally, we make extensive use of Theorem 3.2 when possible and the highest weight vector strategy from Theorem 4.14. The simple $\mathfrak{sl}(4)$ modules are given by the restricted dominant integral weights of which there are $3^3 = 27$. We again use the decomposition algorithm in 5.1 to decompose each $L(\lambda)$ into Jordan blocks (w.r.t e_{34}) to help determine the semisimplification. The relevant decompositions and actions maps are described in Appendix A.2 while the semisimplified module structure is described below.

1. $\lambda = (2, 2, 2, 2)$
   We have $(L(\lambda), e_{34}) = J_1$ which means the semisimplification is $(1|0)$-dimensional. The highest weight vector has semisimplified weight $\overline{\lambda} = (2, 2, 4)$ and by virtue of the semisimplification being $(1|0)$-dimensional it is obviously simple. Therefore, we can conclude that $\overline{L(\lambda)} = L(2, 2, 4)$.
2. $\lambda = (2, 2, 2, 1)$
   We have $(L(\lambda), e_{34}) = 2J_1 \oplus J_2$ which means the semisimplification is $(2|1)$-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\overline{\lambda} = (2, 2, 3)$. Using Theorem 3.2 tells us the dimension of the module generated by $\overline{\lambda}$ must be 3 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(2, 2, 3)$.
3. $\lambda = (2, 2, 2, 0)$
   We have $(L(\lambda), e_{34}) = 3J_1 \oplus 2J_2 \oplus J_3$ which means the semisimplification is $(3|2)$-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\lambda' = (2, 1, 3)$. Using Theorem 3.2 tells us that the dimension of the module generated by λ' is 5 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(2, 1, 3)$.
4. $\lambda = (2, 2, 1, 1)$
   We have $(L(\lambda), e_{34}) = 2J_1 \oplus 2J_2$ which means the semisimplification is $(2|2)$-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\overline{\lambda} = (2, 2, 2)$. Using Theorem 3.2 tells us the dimension of the module generated by $\overline{\lambda}$ is 4 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(2, 2, 2)$.
5. $\lambda = (2, 2, 1, 0)$
   We have $(L(\lambda), e_{34}) = 2J_1 \oplus 4J_2 \oplus 2J_3$ which means the semisimplification is $(2|4)$-dimensional. Using the algorithm we can see two highest weight vectors with semisimplified weights $\lambda_1 = (2, 2, 1)$ and $\lambda_2 = (2, 0, 3)$. Using Theorem 3.2 tells us that $\dim L(\lambda_1) = 1$ and $\dim L(\lambda_2) = 5$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2, 2, 1) \oplus L(2, 0, 3)$.
6. $\lambda = (2, 2, 1, -1)$
   We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 9J_3$ which means the semisimplification is $(6|6)$-dimensional. Using the algorithm we see there is a unique highest weight vector with semsimplified weight $\lambda' = (2, 0, 2)$. Using Theorem 3.2 tells us the dimension of the module generated by λ' is 12 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(2, 0, 2)$.
7. $\lambda = (2, 2, 0, 0)$
   We have $(L(\lambda), e_{34}) = 2J_1 \oplus 4J_2 \oplus 3J_3$ which means the semisimplification is $(2|4)$-dimensional. Using the algorithm we can see two highest weight vectors with semisimplified weights $\lambda_1 = (2, 2, 0)$ and $\lambda_2 = (1, 0, 3)$. Using Theorem 3.2 tells us that

dim $L(\lambda_1) = 3$ and dim $L(\lambda_2) = 3$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2,2,0) \oplus L(1,0,3)$.

8. $\lambda = (2,2,0,-1)$
We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 14J_3$ which means the semisimplification is $(6|6)$-dimensional. Using the algorithm we can see two highest weight vectors with semisimplified weights $\lambda_1 = (2,2,-1)$ and $\lambda_2 = (1,0,2)$. Using Theorem 3.2 tells us that dim $L(\lambda_1) = 4$ and dim $L(\lambda_2) = 8$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2,2,-1) \oplus L(1,0,2)$.
9. $\lambda = (2,2,0,-2)$
We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 36J_3$ which means the semisimplification is $(6|6)$-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\lambda' = (1,-1,2)$. Using Theorem 3.2 tells us the dimension of the module generated by λ' is 12 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(1,-1,2)$.
10. $\lambda = (2,1,1,1)$
We have $(L(\lambda), e_{34}) = 2J_1 \oplus J_2$ which means the semisimplification is $(2|1)$-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\overline{\lambda} = (2,1,2)$. Using Theorem 3.2 tells us the dimension of the module generated by $\overline{\lambda}$ is 3 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(2,1,2)$.
11. $\lambda = (2,1,1,0)$
We have $(L(\lambda), e_{34}) = 4J_1 \oplus 4J_2 \oplus J_3$ which means the semisimplification is $(4|4)$-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\overline{\lambda} = (2,1,1)$. Using Theorem 3.2 tells us the dimension of the module generated by $\overline{\lambda}$ is 8 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(2,1,1)$
12. $\lambda = (2,1,1,-1)$
We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 6J_3$ which means the semisimplification is $(6|6)$-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\lambda' = (2,0,1)$. Using Theorem 3.2 tells us the dimension of the module generated by λ' is 12 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(2,0,1)$.
13. $\lambda = (2,1,0,0)$
We have $(L(\lambda), e_{34}) = 2J_1 \oplus 4J_2 \oplus 2J_3$ which means the semisimplification is $(2|4)$-dimensional. Using the algorithm we can see two highest weight vectors with semisimplified weights $\lambda_1 = (2,1,0)$ and $\lambda_2 = (0,0,3)$. Using Theorem 3.2 tells us that dim $L(\lambda_1) = 5$ and dim $L(\lambda_2) = 1$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2,1,0) \oplus L(0,0,3)$.
14. $\lambda = (2,1,0,-1)$
We have $(L(\lambda), e_{34}) = 2J_1 \oplus 6J_2 \oplus 10J_3$ which means the semisimplification is $(2|6)$-dimensional. Using the algorithm we can see four highest weight vectors with semisimplified weights $\lambda_1 = (2,1,-1)$, $\lambda_2 = (-1,2,1)$, $\lambda_3 = (2,-1,1)$ and $\lambda_4 = (0,0,2)$. Using Theorem 3.2 tells us that dim $L(\lambda_1) = 3$, dim $L(\lambda_2) = 1$, dim $L(\lambda_3) = 1$ and dim $L(\lambda_4) = 3$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2,1,-1) \oplus L(-1,2,1) \oplus L(2,-1,1) \oplus L(0,0,2)$.
15. $\lambda = (2,1,0,-2)$
We have $(L(\lambda), e_{34}) = 4J_1 \oplus 8J_2 \oplus 32J_3$ which means the semisimplification is

(4|8)-dimensional. Using the algorithm we can see four highest weight vectors with semisimplified weights $\lambda_1 = (1,1,-1)$, $\lambda_2 = (2,-1,0)$, $\lambda_3 = (-1,2,0)$, and $\lambda_4 = (0,-1,2)$. Using Theorem 3.2 tells us that dim $L(\lambda_1) = 1$, dim $L(\lambda_2) = 3$, dim $L(\lambda_3) = 3$ and dim $L(\lambda_4) = 5$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(1,1,-1) \oplus L(2,-1,0) \oplus L(-1,2,0) \oplus L(0,-1,2)$.

16. $\lambda = (2,1,-1,-1)$
We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 14J_3$ which means the semisimplification is (6|6)-dimensional. Using the algorithm we can see two highest weight vectors with semisimplified weights $\lambda_1 = (2,1,-2)$ and $\lambda_2 = (0,0,1)$. Using Theorem 3.2 tells us that dim $L(\lambda_1) = 8$ and dim $L(\lambda_2) = 4$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2,1,-2) \oplus L(0,0,1)$.
17. $\lambda = (2,1,-1,-2)$
We have $(L(\lambda), e_{34}) = 7J_1 \oplus 4J_2 \oplus 47J_3$ which means the semisimplification is (7|4)-dimensional. Using the algorithm we can see three highest weight vectors with semisimplified weights $\lambda_1 = (2,1,-3)$, $\lambda_2 = (0,0,0)$, $\lambda_3 = (0,-2,2)$. Using Theorem 3.2 tells us that dim $L(\lambda_1) = 5$, dim $L(\lambda_2) = 1$ and dim $L(\lambda_3) = 5$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2,1,-3) \oplus L(0,0,0) \oplus L(0,-2,2)$.
18. $\lambda = (2,1,-1,-3)$
We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 114J_3$ which means the semisimplification is (6|6)-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\lambda' = (0,-2,1)$. Using Theorem 3.2 tells us the dimension of the module generated by λ' is 12 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(0,-2,1)$.
19. $\lambda = (2,0,0,0)$
This is $S^2(V)$ and must be simple. In particular, we have $\overline{L(\lambda)} = L(2,0,0)$.
20. $\lambda = (2,0,0,-1)$
We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 6J_3$ which means the semisimplification is (6|6)-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\overline{\lambda} = (2,0,-1)$. Using Theorem 3.2 tells us the dimension of the module generated by $\overline{\lambda}$ is 12 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(2,0,-1)$.
21. $\lambda = (2,0,0,-2)$
We have $(L(\lambda), e_{34}) = 4J_1 \oplus 4J_2 \oplus 19J_3$ which means the semisimplification is (4|4)-dimensional. Using the algorithm we can see two highest weight vectors with semisimplified weights $\lambda_1 = (2,-1,-1)$ and $\lambda_2 = (-1,2,-1)$. Using Theorem 3.2 tells us that dim $L(\lambda_1) = 4$ and dim $L(\lambda_2) = 4$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2,-1,-1) \oplus L(-1,2,-1)$.
22. $\lambda = (2,0,-1,-1)$
We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 9J_3$ which means the semisimplification is (6|6)-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\overline{\lambda} = (2,0,-2)$. Using Theorem 3.2 tells us the dimension of the module generated by $\overline{\lambda}$ is 12 as needed. Therefore, we can conclude that $\overline{L(\lambda)} = L(2,0,-2)$.

23. $\lambda = (2, 0, -1, -2)$
We have $(L(\lambda), e_{34}) = 4J_1 \oplus 8J_2 \oplus 32J_3$ which means the semisimplification is (4|8)-dimensional. Using the algorithm we can see four highest weight vectors with semisimplified weights $\lambda_1 = (2, 0, -3)$, $\lambda_2 = (-1, 1, -1)$, $\lambda_3 = (-1, -1, 1)$, $\lambda_4 = (2, -2, -1)$. Using Theorem 3.2 tells us that dim $L(\lambda_1) = 5$, dim $L(\lambda_2) = 3$, dim $L(\lambda_3) = 1$, and dim $L(\lambda_4) = 3$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2, 0, -3) \oplus L(-1, 1, -1) \oplus L(-1, -1, 1) \oplus L(2, -2, -1)$.
24. $\lambda = (2, 0, -1, -3)$
We have $(L(\lambda), e_{34}) = 8J_1 \oplus 8J_2 \oplus 90J_3$ which means the semisimplification is (8|8)-dimensional. Using the algorithm we can see two highest weight vectors with semisimplified weights $\lambda_1 = (2, -2, -2)$ and $\lambda_2 = (-1, 1, -2)$. Using Theorem 3.2 tells us that dim $L(\lambda_1) = 8$ and dim $L(\lambda_2) = 8$. Combining this with Theorem 4.14 allows us to conclude that $\overline{L(\lambda)} = L(2, -2, -2) \oplus L(-1, 1, -2)$.
25. $\lambda = (2, 0, -2, -2)$
We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 36J_3$ which means the semisimplification is (6|6)-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\overline{\lambda} = (2, 0, -4)$. Using Theorem 3.2 tells us the dimension of the module generated by $\overline{\lambda}$ is 12. Therefore, we can conclude that $\overline{L(\lambda)} = L(2, 0, -4)$.
26. $\lambda = (2, 0, -2, -3)$
We have $(L(\lambda), e_{34}) = 6J_1 \oplus 6J_2 \oplus 114J_3$ which means the semisimplification is (6|6)-dimensional. Using the algorithm we see there is a unique highest weight vector with semisimplified weight $\overline{\lambda} = (2, 0, -5)$. Using Theorem 3.2 tells us the dimension of the module generated by $\overline{\lambda}$ is 12, Therefore, we can conclude that $\overline{L(\lambda)} = L(2, 0, -5)$.
27. $\lambda = (2, 0, -2, -4)$
We have $(L(\lambda), e_{34}) = 243J_3$ which means the semisimplification is the zero-map.

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Appendix A. Action Maps

GAP Generator	Matrix Element
x_1	e_{12}
x_2	e_{23}
x_3	e_{13}
y_1	e_{21}
y_2	e_{32}
y_3	e_{31}
h_7	$e_{11} - e_{22}$
h_8	$e_{22} - e_{33}$

Table 2. Standard Basis Generators for $\mathfrak{sl}_3$

A.1. **Semisimplifications of $\mathfrak{gl}(3)$-modules in characteristic** 3. As seen in Table 2, the 8-dimensional Lie algebra $\mathfrak{sl}(3)$, of type A_2, is built from two simple root vectors (x_1, x_2) and their negatives (y_1, y_2). The Lie brackets among these generates the remaining positive root (x_3) and negative roots (y_3). The Cartan subalgebra is 2-dimensional, spanned by $h_7 = e_{11} - e_{22}$, and $h_8 = e_{22} - e_{33}$. Under the action of e_{32}, $\mathfrak{sl}(3)$ decomposes as $J_1 \oplus 2J_2 \oplus J_3$ where the copy of J_1 is $h_7 - h_8$ and the copies of J_2 are $y_1 \to -y_3$ and $x_3 \to x_1$. We now enumerate the action maps for each of the 8 restricted integral weights. Since J_3 vanishes after semisimplification we omit listing the J_3 modules in the decomposition.

1. Dominant Weight: $\lambda = [2, 2, 2]$

- Weyl module : $V[0, 0]$ of dimension 1.
- $L(\lambda)$ is a 1−dimensional quotient of $V[0, 0]$.

Decomposition of $L(\lambda)$ under action of $e_{32} : J_1$

$$J_1 : v_0 \ (u_1)$$

Action map

	u_1
$h_7 - h_8$	0
$y_1 \to -y_3$	0
$x_3 \to x_1$	0

2. Dominant Weight: $\lambda = [2, 2, 1]$

- Weyl module : $V[0, 1]$ of dimension 3.
- $L(\lambda)$ is a 3−dimensional quotient of $V[0, 1]$.

Decomposition of $L(\lambda)$ under action of $e_{32} : J_1 \oplus J_2$

$$J_1 : y_3 v_0 \ (u_1)$$

$$J_2 : v_0 \to y_2 v_0 \ (w_1)$$

Action map

	u_1	w_1
$h_7 - h_8$	$2u_1$	$2w_1$
$y_1 \to -y_3$	0	$2u_1$
$x_3 \to x_1$	w_1	0

3. Dominant Weight: $\lambda = [2, 2, 0]$

- Weyl module : $V[0, 2]$ of dimension 6.
- $L(\lambda)$ is a 6−dimensional quotient of $V[0, 2]$.

Decomposition of $L(\lambda)$ under action of $e_{32} : J_1 \oplus J_2 \oplus J_3$

$$J_1 : y_3^{(2)} v_0 \ (u_1)$$
$$J_2 : y_3 v_0 \to y_2 y_3 v_0 \ (w_1)$$

Action map

	u_1	w_1
$h_7 - h_8$	u_1	w_1
$y_1 \to -y_3$	0	u_1
$x_3 \to x_1$	w_1	0

4. Dominant Weight: $\lambda = [2, 1, 1]$

- Weyl module : $V[1, 0]$ of dimension 3.
- $L(\lambda)$ is a 3−dimensional quotient of $V[1, 0]$.

Decomposition of $L(\lambda)$ under action of $e_{32} : J_1 \oplus J_2$

$$J_1 : v_0 \ (u_1)$$
$$J_2 : y_1 v_0 \to 2y_3 v_0 \ (w_1)$$

Action map

	u_1	w_1
$h_7 - h_8$	u_1	w_1
$y_1 \to -y_3$	w_1	0
$x_3 \to x_1$	0	u_1

5. Dominant Weight: $\lambda = [2, 1, 0]$

- Weyl module : $V[1, 1]$ of dimension 8.
- $L(\lambda)$ is a 7−dimensional quotient of $V[1, 1]$.

Decomposition of $L(\lambda)$ under action of $e_{32} : 2J_2 \oplus J_3$

$$J_2 : v_0 \to y_2 v_0 \ (w_1)$$

$$J_2 : y_1 y_3 v_0 \to 2y_3^{(2)} v_0 \ (w_2)$$

Action map

	w_1	w_1
$h_7 - h_8$	0	0
$y_1 \to -y_3$	0	0
$x_3 \to x_1$	0	0

6. Dominant Weight: $\lambda = [2, 1, -1]$

- Weyl module : $V[1, 2]$ of dimension 15.
- $L(\lambda)$ is a 15−dimensional quotient of $V[1, 2]$.

Decomposition of $L(\lambda)$ under action of $e_{32} : J_1 \oplus J_2 \oplus 4J_3$

$$J_1 : y_1 y_2 y_3 v_0 \ (u_1)$$

$$J_2 : y_1 y_3^{(2)} v_0 \to 2y_3^{(3)} v_0 \ (w_1)$$

Action map

	u_1	w_1
$h_7 - h_8$	$2u_1$	0
$y_1 \to -y_3$	w_1	0
$x_3 \to x_1$	0	$2u_1$

7. Dominant Weight: $\lambda = [2, 0, 0]$

- Weyl module : $V[2, 0]$ of dimension 6.
- $L(\lambda)$ is a 6−dimensional quotient of $V[2, 0]$.

Decomposition of $L(\lambda)$ under action of $e_{32} : J_1 \oplus J_2 \oplus J_3$

$$J_1 : v_0 \ (u_1)$$

$$J_2 : y_1 v_0 \to 2y_3 v_0 \ (w_1)$$

Action map

	u_1	w_1
$h_7 - h_8$	$2u_1$	$2w_1$
$y_1 \to -y_3$	w_1	0
$x_3 \to x_1$	0	$2u_1$

8. Dominant Weight: $\lambda = [2, 0, -1]$

- Weyl module : $V[2, 1]$ of dimension 15.
- $L(\lambda)$ is a 15−dimensional quotient of $V[2, 1]$.

Decomposition of $L(\lambda)$ under action of $e_{32} : J_1 \oplus J_2 \oplus 4J_3$

$$J_1 : y_1y_2v_0 + y_3v_0 \ (u_1)$$
$$J_2 : v_0 \to y_2v_0 \ (w_1)$$

Action map

	u_1	w_1
$h_7 - h_8$	u_1	w_1
$y_1 \to -y_3$	0	u_1
$x_3 \to x_1$	w_1	0

9. Dominant Weight: $\lambda = [2, 0, -2]$

- Weyl module : $V[2, 2]$ of dimension 27.
- $L(\lambda)$ is a 27−dimensional quotient of $V[2, 2]$.

Decomposition of $L(\lambda)$ under action of $e_{32} : 9J_3$
Semisimplification is the zero map as all J_3 vanish after semisimplification.

GAP Generator	Matrix Element
x_1	e_{12}
x_2	e_{23}
x_3	e_{34}
x_4	e_{13}
x_5	e_{24}
x_6	e_{14}
y_1	e_{21}
y_2	e_{32}
y_3	e_{43}
y_4	e_{31}
y_5	e_{42}
y_6	e_{41}
h_{13}	$e_{11} - e_{22}$
h_{14}	$e_{22} - e_{33}$
h_{15}	$e_{33} - e_{44}$

TABLE 3. Standard Basis Generators for $\mathfrak{sl}_4$

A.2. **Semisimplifications of $\mathfrak{gl}(4)$-modules in characteristic 3.** As seen in Table 3, the 15-dimensional Lie algebra $\mathfrak{sl}(4)$, of type A_3, is built from three simple root vectors

(x_1, x_2, x_3) and their negatives (y_1, y_2, y_3). The Lie brackets among these generates the remaining positive roots (x_4, x_5, x_6) and negative roots (y_4, y_5, y_6). The Cartan subalgebra is 3-dimensional, spanned by $h_{13} = e_{11} - e_{22}$, $h_{14} = e_{22} - e_{33}$, and $h_{15} = e_{33} - e_{44}$. Under the action of e_{43}, $\mathfrak{sl}(4)$ decomposes as $4J_1 \oplus 4J_2 \oplus J_3$ where the copies of J_1 are $x_1, y_1, h_{13} - h_{14} + h_{15}, h_{14} - h_{15}$ and the copies of J_2 are $y_4 \to -y_6$, $y_2 \to -y_5$, $x_5 \to x_2$ and $x_6 \to x_4$. We now enumerate the action maps for the key subcases for the 27 restricted integral weights. Since J_3 vanishes after semisimplification we omit listing the J_3 modules in the decomposition.

1. Dominant Weight: $\lambda = [2, 2, 2, 2]$

- Weyl module, $V[0, 0, 0]$ Dim=1
- $L(\lambda)$ is 1-dimensional quotient of V[0, 0, 0]

Decomposition of $L(\lambda)$ under action of $e_{43} : J_1$

$$J_1 : v_0 \ (u_1)$$

Action map

	u_1
y_1	0
x_1	0
$h_{13} - h_{14} + h_{15}$	0
$h_{14} - h_{15}$	0
$y_4 \to -y_6$	0
$y_2 \to -y_5$	0
$x_5 \to x_2$	0
$x_6 \to x_4$	0

2. Dominant Weight: $\lambda = [2, 2, 2, 1]$

- Weyl module $V[0, 0, 1]$ Dim=4
- $L(\lambda)$ is 4-dimensional quotient of V[0, 0, 1]

Decomposition of $L(\lambda)$ under action of $e_{43} : 2J_1 \oplus J_2$

$$J_1 : y_5 v_0 \ (u_1)$$
$$J_1 : y_6 v_0 \ (u_2)$$
$$J_2 : v_0 \to y_3 v_0 (w_1)$$

Action map

	u_1	u_2	w_1
y_1	u_2	0	0
x_1	0	u_1	0
$h_{13}-h_{14}+h_{15}$	$2u_1$	$2u_2$	w_1
$h_{14}-h_{15}$	$2u_1$	0	$2w_1$
$y_4 \to -y_6$	0	0	u_2
$y_2 \to -y_5$	0	0	u_1
$x_5 \to x_2$	w_1	0	0
$x_6 \to x_4$	0	w_1	0

3. Dominant Weight: $\lambda = [2,2,2,0]$

- Weyl module $V[0,0,2]$ Dim=10
- $L(\lambda)$ is 10-dimensional quotient of $V[0,0,2]$

Decomposition of $L(\lambda)$ under action of $e_{43} : 3J_1 \oplus 2J_2 \oplus J_3$

$$J_1 : y_5^{(2)} v_0 \ (u_1)$$
$$J_1 : y_5 y_6 v_0 \ (u_2)$$
$$J_1 : y_6^{(2)} v_0 \ (u_3)$$
$$J_2 : y_5 v_0 \to 2y_3 y_5 v_0 \ (w_1)$$
$$J_2 : y_6 v_0 \to 2y_3 y_6 v_0 \ (w_2)$$

Action map

	u_1	u_2	u_3	w_1	w_2
y_1	u_2	$2u_3$	0	w_2	0
x_1	0	$2u_1$	u_2	0	w_1
$h_{14}-h_{15}$	u_1	$2u_2$	0	w_1	$2w_2$
$h_{13}-h_{14}+h_{15}$	u_1	u_2	u_3	0	0
$y_4 \to -y_6$	0	0	0	u_2	$2u_3$
$y_2 \to -y_5$	0	0	0	$2u_1$	u_2
$x_5 \to x_2$	w_1	w_2	0	0	0
$x_6 \to x_4$	0	w_1	w_2	0	0

4. Dominant Weight: $\lambda = [2,2,1,1]$

- Weyl module : $V[0,1,0]$ of dimension 6.
- $L(\lambda)$ is a $6-$dimensional quotient of $V[0,1,0]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 2J_1 \oplus 2J_2$

$$J_1 : v_0 \ (u_1)$$
$$J_1 : y_2 y_6 v_0 \ (u_2)$$
$$J_2 : y_2 v_0 \to 2y_5 v_0 \ (w_1)$$
$$J_2 : y_4 v_0 \to y_6 v_0 \ (w_2)$$

Action map

	u_1	u_2	w_1	w_2
y_1	0	0	w_2	0
x_1	0	0	0	w_1
$h_{14} - h_{15}$	u_1	$2u_2$	w_1	$2w_2$
$h_{13} - h_{14} + h_{15}$	$2u_1$	u_2	0	0
$y_2 \to -y_5$	w_1	0	0	$2u_2$
$y_4 \to -y_6$	w_2	0	u_2	0
$x_5 \to x_2$	0	$2w_2$	$2u_1$	0
$x_6 \to x_4$	0	w_1	0	$2u_1$

5. Dominant Weight: $\lambda = [2, 2, 1, 0]$

- Weyl module : $V[0, 1, 1]$ of dimension 20.
- $L(\lambda)$ is a 16−dimensional quotient of $V[0, 1, 1]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 2J_1 \oplus 4J_2 \oplus 2J_3$

$$J_1 : y_2y_5y_6v_0 \ (u_1)$$
$$J_1 : y_2y_6^{(2)}v_0 \ (u_2)$$
$$J_2 : v_0 \to 2y_3v_0 \ (w_1)$$
$$J_2 : y_2y_5v_0 \to 2y_5^{(2)}v_0 \ (w_2)$$
$$J_2 : y_2y_6v_0 \to 2y_2y_3y_6v_0 \ (w_3)$$
$$J_2 : y_4y_6v_0 \to 2y_6^{(2)}v_0 \ (w_4)$$

Action map

	u_1	u_2	w_1	w_2	w_3	w_4
y_1	u_2	0	0	$2w_3$	w_4	0
x_1	0	u_1	0	0	w_2	$2w_3$
$h_{14} - h_{15}$	u_1	$2u_2$	0	0	w_3	$2w_4$
$h_{13} - h_{14} + h_{15}$	0	0	0	$2w_2$	$2w_3$	$2w_4$
$y_2 \downarrow y_5$	0	0	0	0	u_1	$2u_2$
$y_4 \downarrow y_6$	0	0	0	u_1	$2u_2$	0
$x_5 \downarrow -x_2$	$2w_3$	$2w_4$	0	0	0	0
$x_6 \downarrow -x_4$	w_2	w_3	0	0	0	0

6. Dominant Weight: $\lambda = [2, 2, 1, -1]$

- Weyl module : $V[0, 1, 2]$ of dimension 45.
- $L(\lambda)$ is a 45−dimensional quotient of $V[0, 1, 2]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 6J_1 \oplus 6J_2 \oplus 9J_3$

$$J_1 : y_2y_3y_5v_0 \ (u_1)$$
$$J_1 : y_2y_3y_6v_0 + y_3y_4y_5v_0 + y_5y_6v_0 \ (u_2)$$
$$J_1 : y_3y_4y_6v_0 + 2y_6^{(2)}v_0 \ (u_3)$$
$$J_1 : y_2y_5^{(2)}y_6v_0 \ (u_4)$$
$$J_1 : y_2y_5y_6^{(2)}v_0 \ (u_5)$$
$$J_1 : y_2y_6^{(3)}v_0 \ (u_6)$$
$$J_2 : y_2y_5^{(2)}v_0 \to 2y_5^{(3)}v_0 \ (w_1)$$
$$J_2 : y_2y_5y_6v_0 \to y_2y_3y_5y_6v_0 + y_5^{(2)}y_6v_0 \ (w_2)$$
$$J_2 : y_4y_5^{(2)}v_0 \to 2y_2y_3y_5y_6v_0 + y_5^{(2)}y_6v_0 \ (w_3)$$
$$J_2 : y_2y_6^{(2)}v_0 \to y_2y_3y_6^{(2)}v_0 + 2y_5y_6^{(2)}v_0 \ (w_4)$$
$$J_2 : y_4y_5y_6v_0 \to 2y_2y_3y_6^{(2)}v_0 \ (w_5)$$
$$J_2 : y_4y_6^{(2)}v_0 \to 2y_6^{(3)}v_0 \ (w_6)$$

Action map

	u_1	u_2	u_3	u_4	u_5	u_6	w_1	w_2	w_3	w_4	w_5	w_6
y_1	u_2	$2u_3$	0	u_5	$2u_6$	0	$w_2 + w_3$	$2w_4 + w_5$	w_5	w_6	$2w_6$	0
x_1	0	$2u_1$	u_2	0	$2u_4$	u_5	0	$2w_1$	w_1	w_2	$w_2 + 2w_3$	$w_4 + w_5$
$h_{14} - h_{15}$	$2u_1$	0	u_3	0	u_5	$2u_6$	$2w_1$	0	0	w_4	w_5	$2w_6$
$h_{13} - h_{14} + h_{15}$	0	0	0	$2u_4$	$2u_5$	$2u_6$	w_1	w_2	w_3	w_4	w_5	w_6
$y_2 \to -y_5$	w_1	$2w_3$	$2w_4 + w_5$	0	0	0	0	u_4	0	u_5	u_5	$2u_6$
$y_4 \to -y_6$	$w_2 + 2w_3$	$2w_4$	w_6	0	0	0	u_4	$2u_5$	$2u_5$	0	u_6	0
$x_5 \to x_2$	0	0	0	$2w_3$	$2w_5$	$2w_6$	u_1	u_2	0	u_3	0	0
$x_6 \to x_4$	0	0	0	w_1	w_2	w_4	0	0	u_1	0	u_2	u_3

7. Dominant Weight: $\lambda = [2, 2, 0, 0]$

- Weyl module : $V[0, 2, 0]$ of dimension 20.
- $L(\lambda)$ is a 19−dimensional quotient of $V[0, 2, 0]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 2J_1 \oplus 4J_2 \oplus 3J_3$.

$$J_1 : v_0 \ (u_1)$$
$$J_1 : y_2^{(2)}y_6^{(2)}v_0 \ (u_2)$$
$$J_2 : y_2v_0 \to 2y_5v_0 \ (w_1)$$

$$J_2 : y_4v_0 \to 2y_6v_0 \ (w_2)$$
$$J_2 : y_2^{(2)}y_6v_0 \to 2y_2y_5y_6v_0 \ (w_3)$$
$$J_2 : y_2y_4y_6v_0 \to 2y_2y_6^{(2)}v_0 \ (w_4)$$

Action map

	u_1	u_2	w_1	w_2	w_3	w_4
y_1	0	0	w_2	0	w_4	0
x_1	0	0	0	w_1	0	w_3
$h_{14}-h_{15}$	$2u_1$	u_2	$2w_1$	0	0	w_4
$h_{13}-h_{14}+h_{15}$	u_1	$2u_2$	$2w_1$	$2w_2$	w_3	w_4
$y_2 \to -y_5$	w_1	0	0	0	0	u_2
$y_4 \to -y_6$	w_2	0	0	0	$2u_2$	0
$x_5 \to x_2$	0	$2w_4$	u_1	0	0	0
$x_6 \to x_4$	0	w_3	0	$1v_0$	0	0

8. Dominant Weight: $\lambda = [2,2,0,-1]$

- Weyl module : $V[0,2,1]$ of dimension 60.
- $L(\lambda)$ is a 60−dimensional quotient of $V[0,2,1]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 6J_1 \oplus 6J_2 \oplus 14J_3$.

$$J_1 : y_2y_3v_0 + y_5v_0 \ (u_1)$$
$$J_1 : y_3y_4v_0 + 2y_6v_0 \ (u_2)$$
$$J_1 : y_2^{(2)}y_3y_6v_0 + 2y_4y_5^{(2)}v_0 \ (u_3)$$
$$J_1 : 2y_2y_3y_4y_6v_0 + y_2y_6^{(2)}v_0 + y_4y_5y_6v_0 \ (u_4)$$
$$J_1 : y_2^{(2)}y_5y_6^{(2)}v_0 \ (u_5)$$
$$J_1 : y_2^{(2)}y_6^{(3)}v_0 \ (u_6)$$
$$J_2 : v_0 \to y_3v_0 \ (w_1)$$
$$J_2 : y_2y_3y_4v_0 + 2y_2y_6v_0 \to 2y_3y_4y_5v_0 + y_5y_6v_0 \ (w_2)$$
$$J_2 : y_2^{(2)}y_5y_6v_0 \to 2y_2y_5^{(2)}y_6v_0 \ (w_3)$$
$$J_2 : y_2y_4y_5y_6v_0 \to y_2y_5y_6^{(2)}v_0 + y_2^{(2)}y_3y_6^{(2)}v_0 \ (w_4)$$
$$J_2 : y_2^{(2)}y_6^{(2)}v_0 \to 2y_2y_5y_6^{(2)}v_0 + y_2^{(2)}y_3y_6^{(2)}v_0 \ (w_5)$$
$$J_2 : y_2y_4y_6^{(2)}v_0 \to 2y_2y_6^{(3)}v_0 \ (w_6)$$

Action map

	u_1	u_2	u_3	u_4	u_5	u_6	w_1	w_2	w_3	w_4	w_5	w_6
y_1	u_2	0	$2u_4$	0	u_6	0	0	0	w_4+2w_5	0	w_6	0
x_1	0	u_1	0	$2u_3$	0	u_5	0	0	0	0	w_3	w_4+2w_5
$h_{14}-h_{15}$	u_1	$2u_2$	$2u_3$	0	0	u_6	w_1	$2w_2$	$2w_3$	0	0	w_6
$h_{13}-h_{14}+h_{15}$	0	0	$2u_3$	$2u_4$	u_5	u_6	w_1	w_2	0	0	0	0
$y_2\to -y_5$	0	w_2	w_3	$2w_4+2w_5$	0	0	$2u_1$	0	0	$2u_5$	u_5	u_6
$y_4\to -y_6$	$2w_2$	0	w_5	$2w_6$	0	0	$2u_2$	0	$2u_5$	$2u_6$	0	0
$x_5\to x_2$	w_1	0	$2w_2$	0	$2w_4+2w_5$	$2w_6$	0	u_2	u_3	u_4	$2u_4$	0
$x_6\to x_4$	0	w_1	0	0	w_3	w_5	0	$2u_1$	0	u_3	0	$2u_4$

9. Dominant Weight: $\lambda = [2,2,0,-2]$

- Weyl module : $V[0,2,2]$ of dimension 126.
- $L(\lambda)$ is a 126−dimensional quotient of $V[0,2,2]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 6J_1 \oplus 6J_2 \oplus 36J_3$.

$$J_1 : 2y_2y_5^{(2)}y_6v_0 + y_2^{(2)}y_3y_5y_6v_0 + 2y_4y_5^{(3)}v_0 \ (u_1)$$

$$J_1 : y_2y_3y_4y_5y_6v_0 + 2y_2^{(2)}y_3y_6^{(2)}v_0 + y_4y_5^{(2)}y_6v_0 \ (u_2)$$

$$J_1 : 2y_2y_3y_4y_6^{(2)}v_0 + y_2y_6^{(3)}v_0 + y_4y_5y_6^{(2)}v_0 \ (u_3)$$

$$J_1 : y_2^{(2)}y_5^{(2)}y_6^{(2)}v_0 \ (u_4)$$

$$J_1 : y_2^{(2)}y_5y_6^{(3)}v_0 \ (u_5)$$

$$J_1 : y_2^{(2)}y_6^{(4)}v_0 \ (u_6)$$

$$J_2 : y_2^{(2)}y_5^{(2)}y_6v_0 \to 2y_2y_5^{(3)}y_6v_0 \ (w_1)$$

$$J_2 : y_2y_4y_5^{(2)}y_6v_0 \to y_2^{(2)}y_3y_5y_6^{(2)}v_0 \ (w_2)$$

$$J_2 : y_2^{(2)}y_5y_6^{(2)}v_0 \to y_2y_5^{(2)}y_6^{(2)}v_0 + y_2^{(2)}y_3y_5y_6^{(2)}v_0 \ (w_3)$$

$$J_2 : y_2y_4y_5y_6^{(2)}v_0 \to y_2y_5y_6^{(3)}v_0 + y_2^{(2)}y_3y_6^{(3)}v_0 \ (w_4)$$

$$J_2 : y_2^{(2)}y_6^{(3)}v_0 \to 2y_2y_5y_6^{(3)}v_0 + y_2^{(2)}y_3y_6^{(3)}v_0 \ (w_5)$$

$$J_2 : y_2y_4y_6^{(3)}v_0 \to 2y_2y_6^{(4)}v_0 \ (w_6)$$

Action map

	u_1	u_2	u_3	u_4	u_5	u_6	w_1	w_2	w_3	w_4	w_5	w_6
y_1	u_2	u_3	0	u_5	$2u_6$	0	w_2+w_3	w_5	w_4	w_6	w_6	0
x_1	0	$2u_1$	$2u_2$	0	$2u_4$	u_5	0	$2w_1$	$2w_1$	w_2+w_3	w_3	w_4+2w_5
$h_{14}-h_{15}$	u_1	$2u_2$	0	$2u_4$	0	u_6	w_1	$2w_2$	$2w_3$	0	0	w_6
$h_{13}-h_{14}+h_{15}$	u_1	u_2	u_3	0	0	0	$2w_1$	$2w_2$	$2w_3$	$2w_4$	$2w_5$	$2w_6$
$y_2\to -y_5$	$2w_1$	$2w_2$	$2w_4+w_5$	0	0	0	0	0	$2u_4$	$2u_5$	u_5	u_6
$y_4\to -y_6$	w_3	w_4	w_6	0	0	0	$2u_4$	$2u_5$	0	0	u_6	0
$x_5\to x_2$	0	0	0	$2w_2+2w_3$	$2w_4+w_5$	$2w_6$	u_1	$2u_2$	u_2	u_3	$2u_3$	0
$x_6\to x_4$	0	0	0	w_1	w_3	w_5	0	u_1	0	u_2	0	$2u_3$

10. Dominant Weight: $\lambda = [2,1,1,1]$

- Weyl module : $V[1,0,0]$ of dimension 4.
- $L(\lambda)$ is a 4−dimensional quotient of $V[1,0,0]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 2J_1 \oplus J_2$.

$$J_1 : v_0 \ (u_1)$$
$$J_1 : y_1v_0 \ (u_2)$$
$$J_2 : y_4v_0 \to 2y_6v_0 \ (w_1)$$

	u_1	u_2	w_1
y_1	u_2	0	0
x_1	0	u_1	0
$h_{14} - h_{15}$	0	u_2	w_1
$h_{13} - h_{14} + h_{15}$	u_1	u_2	$2w_1$
$y_2 \to -y_5$	0	$2w_1$	0
$y_4 \to -y_6$	w_1	0	0
$x_5 \to x_2$	0	0	u_2
$x_6 \to x_4$	0	0	$2u_1$

11. Dominant Weight: $\lambda = [2, 1, 1, 0]$

- Weyl module : $V[1, 0, 1]$ of dimension 15.
- $L(\lambda)$ is a 15−dimensional quotient of $V[1, 0, 1]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 4J_1 \oplus 4J_2 \oplus J_3$.

$$J_1 : y_5v_0 \ (u_1)$$
$$J_1 : y_1y_5v_0 \ (u_2)$$
$$J_1 : y_3y_4v_0 + 2y_6v_0 \ (u_3)$$
$$J_1 : y_1y_6v_0 \ (u_4)$$
$$J_2 : v_0 \to y_3v_0 \ (w_1)$$
$$J_2 : y_1v_0 \to y_1y_3v_0 \ (w_2)$$
$$J_2 : y_4y_5v_0 \to 2y_5y_6v_0 \ (w_3)$$
$$J_2 : y_4y_6v_0 \to 2y_6^{(2)}v_0 \ (w_4)$$

Action map

	u_1	u_2	u_3	u_4	w_1	w_2	w_3	w_4
y_1	u_2	$2u_4$	$2u_4$	0	w_2	0	w_4	0
x_1	0	$2u_1$	$2u_1$	u_2	0	w_1	0	w_3
$h_{14} - h_{15}$	$2u_1$	0	0	u_4	$2w_1$	0	0	w_4
$h_{13} - h_{14} + h_{15}$	0	0	0	0	$2w_1$	$2w_2$	w_3	w_4
$y_2 \to -y_5$	0	$2w_3$	w_3	$2w_4$	u_1	$u_2 + u_3$	0	0
$y_4 \to -y_6$	w_3	w_4	0	0	$2u_3$	u_4	0	0
$x_5 \to x_2$	w_1	w_2	$2w_2$	0	0	0	$u_2 + u_3$	u_4
$x_6 \to x_4$	0	w_1	0	w_2	0	0	$2u_1$	u_3

12. Dominant Weight: $\lambda = [2, 1, 1, -1]$

- Weyl module : $V[1, 0, 2]$ of dimension 36.
- $L(\lambda)$ is a 36−dimensional quotient of $V[1, 0, 2]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 6J_1 \oplus 6J_2 \oplus 6J_3$.

$$J_1 : y_5^{(2)} v_0 \ (u_1)$$
$$J_1 : y_1 y_5^{(2)} v_0 \ (u_2)$$
$$J_1 : y_3 y_4 y_5 v_0 + 2 y_5 y_6 v_0 \ (u_3)$$
$$J_1 : 2 y_1 y_5 y_6 v_0 + y_3 y_4 y_6 v_0 \ (u_4)$$
$$J_1 : y_3 y_4 y_6 v_0 + 2 y_6^{(2)} v_0 \ (u_5)$$
$$J_1 : y_1 y_6^{(2)} v_0 \ (u_6)$$
$$J_2 : y_5 v_0 \to y_3 y_5 v_0 \ (w_1)$$
$$J_2 : y_1 y_5 v_0 \to y_1 y_3 y_5 v_0 \ (w_2)$$
$$J_2 : y_1 y_6 v_0 \to y_1 y_3 y_6 v_0 \ (w_3)$$
$$J_2 : y_4 y_5^{(2)} v_0 \to 2 y_5^{(2)} y_6 v_0 \ (w_4)$$
$$J_2 : y_4 y_5 y_6 v_0 \to 2 y_5 y_6^{(2)} v_0 \ (w_5)$$
$$J_2 : y_4 y_6^{(2)} v_0 \to 2 y_6^{(3)} v_0 \ (w_6)$$

	u_1	u_2	u_3	u_4	u_5	u_6	w_1	w_2	w_3	w_4	w_5	w_6
y_1	u_2	u_1	u_4	$2u_6$	$2u_6$	0	w_2	$2w_3$	0	w_5	$2w_6$	0
x_1	0	0	u_1	$u_2 + u_3$	u_3	$2u_4 + u_5$	0	$2w_1$	w_2	0	$2w_4$	w_5
$h_{14} - h_{15}$	u_1	$2u_2$	$2u_3$	0	0	u_6	w_1	$2w_2$	0	$2w_4$	0	w_6
$h_{13} - h_{14} + h_{15}$	$2u_1$	$2u_2$	$2u_3$	$2u_4$	$2u_5$	$2u_6$	w_1	w_2	w_3	0	0	0
$y_2 \to -y_5$	0	$2w_4$	$2w_4$	$2w_5$	w_5	$2w_6$	$2u_1$	$2u_2 + u_3$	$2u_4 + 2u_5$	0	0	0
$y_4 \to -y_6$	w_4	w_5	0	0	w_6	0	$2u_3$	$2u_4$	$2u_6$	0	0	0
$x_5 \to x_2$	w_1	w_2	$2w_2$	w_3	$2w_3$	0	0	0	0	$u_2 + u_3$	$2u_4 + 2u_5$	u_6
$x_6 \to x_4$	0	w_1	0	$2w_2$	0	w_3	0	0	0	$2u_1$	u_3	u_5

13. Dominant Weight: $\lambda = [2, 1, 0, 0]$

- Weyl module : $V[1, 1, 0]$ of dimension 20.
- $L(\lambda)$ is a 16−dimensional quotient of $V[1, 1, 0]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 2J_1 \oplus 4J_2 \oplus 2J_3$.

$$J_1 : v_0 \ (u_1)$$
$$J_1 : y_1 v_0 \ (u_2)$$
$$J_2 : y_2 v_0 \to 2 y_5 v_0 \ (w_1)$$
$$J_2 : y_1 y_2 v_0 \to 2 y_1 y_5 v_0 \ (w_2)$$
$$J_2 : y_1 y_4 v_0 \to 2 y_1 y_6 v_0 \ (w_3)$$

$$J_2 : y_2y_4y_6v_0 \to 2y_2y_6^{(2)}v_0 \ (w_4)$$

Action map

	u_1	u_2	w_1	w_2	w_3	w_4
y_1	u_2	0	w_2	$2w_3$	0	0
x_1	0	u_1	0	$2w_1$	w_2	0
$h_{14} - h_{15}$	u_1	$2u_2$	w_1	$2w_2$	0	0
$h_{13} - h_{14} + h_{15}$	0	0	w_1	w_2	w_3	0
$y_2 \to -y_5$	w_1	$2w_2$	0	0	0	0
$y_4 \to -y_6$	$2w_2$	w_3	0	0	0	0
$x_5 \to x_2$	0	0	$2u_1$	$2u_2$	0	0
$x_6 \to x_4$	0	0	0	$2u_1$	$2u_2$	0

14. Dominant Weight: $\lambda = [2, 1, 0, -1]$

- Weyl module : $V[1, 1, 1]$ of dimension 64.
- $L(\lambda)$ is a 44−dimensional quotient of $V[1, 1, 1]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 2J_1 \oplus 6J_2 \oplus 10J_3$.

$$J_1 : y_1y_2y_3v_0 + 2y_6v_0 \ (u_1)$$
$$J_1 : 2y_1y_2y_5y_6v_0 + 2y_2y_3y_4y_6v_0 \ (u_2)$$
$$J_2 : v_0 \to y_3v_0 \ (w_1)$$
$$J_2 : y_1v_0 \to y_1y_3v_0 \ (w_2)$$
$$J_2 : y_2y_5v_0 \to 2y_5^{(2)}v_0 \ (w_3)$$
$$J_2 : y_1y_4y_6v_0 \to 2y_1y_6^{(2)}v_0 \ (w_4)$$
$$J_2 : y_2y_4y_5y_6v_0 \to 2y_2y_5y_6^{(2)}v_0 \ (w_5)$$
$$J_2 : y_2y_4y_6^{(2)}v_0 \to 2y_2y_6^{(3)}v_0 \ (w_6)$$

Action Map

	u_1	u_2	w_1	w_2	w_3	w_4	w_5	w_6
y_1	0	0	w_2	0	0	0	w_6	0
x_1	0	0	0	w_1	0	0	0	w_5
$h_{14} - h_{15}$	u_1	$2u_2$	0	w_2	0	0	$2w_5$	0
$h_{13} - h_{14} + h_{15}$	$2u_1$	$2u_2$	w_1	w_2	0	0	$2w_5$	$2w_6$
$y_2 \to -y_5$	0	$2w_5$	0	u_1	0	0	0	0
$y_4 \to -y_6$	0	$2w_6$	$2u_1$	0	0	0	0	0
$x_5 \to x_2$	$2w_2$	0	0	0	0	0	$2u_2$	0
$x_6 \to x_4$	w_1	0	0	0	0	0	0	$2u_2$

15. Dominant Weight: $\lambda = [2, 1, 0, -2]$

- Weyl module : $V[1, 1, 2]$ of dimension 140.
- $L(\lambda)$ is a 116−dimensional quotient of $V[1, 1, 2]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 4J_1 \oplus 8J_2 \oplus 32J_3$.

$$J_1 : y_1y_2y_5y_6^{(2)}v_0 + y_2y_3y_4y_6^{(2)}v_0 + 2y_2y_6^{(3)}v_0 \ (u_1)$$

$$J_1 : 2y_1y_2y_5^{(2)}y_6v_0 + y_2y_3y_4y_5y_6v_0 + y_2y_5y_6^{(2)}v_0 \ (u_2)$$

$$J_1 : y_1y_3y_4y_6v_0 + 2y_1y_6^{(2)}v_0 \ (u_3)$$

$$J_1 : y_2y_3y_5v_0 \ (u_4)$$

$$J_2 : 2y_1y_2y_3v_0 + 2y_6v_0 \to y_1y_2y_3^{(2)}v_0 + y_1y_3y_5v_0 + 2y_3y_6v_0 \ (w_8)$$

$$J_2 : y_2y_5^{(2)}v_0 \to 2y_5^{(3)}v_0 \ (w_7)$$

$$J_2 : y_1y_2y_5^{(2)}v_0 \to 2y_1y_5^{(3)}v_0 \ (w_6)$$

$$J_2 : y_1y_4y_5y_6v_0 \to 2y_1y_5y_6^{(2)}v_0 + 2y_6^{(3)}v_0 \ (w_5)$$

$$J_2 : y_1y_4y_6^{(2)}v_0 \to 2y_1y_6^{(3)}v_0 \ (w_4)$$

$$J_2 : y_2y_4y_5^{(2)}y_6v_0 \to 2y_2y_5^{(2)}y_6^{(2)}v_0 \ (w_3)$$

$$J_2 : y_2y_4y_5y_6^{(2)}v_0 \to 2y_2y_5y_6^{(3)}v_0 \ (w_2)$$

$$J_2 : y_2y_4y_6^{(3)}v_0 \to 2y_2y_6^{(4)}v_0 \ (w_1)$$

Action Map

	u_4	u_3	u_2	u_1	w_8	w_7	w_6	w_5	w_4	w_3	w_2	w_1
y_1	0	0	u_1	0	0	w_6	0	w_4	0	w_2	$2w_1$	0
x_1	0	0	0	u_2	0	0	w_7	0	w_5	0	$2w_3$	w_2
$h_{14} - h_{15}$	$2u_4$	$2u_3$	u_2	$2u_1$	0	$2w_7$	0	$2w_5$	0	w_3	$2w_2$	0
$h_{13} - h_{14} + h_{15}$	u_4	u_3	0	0	0	$2w_7$	$2w_6$	$2w_5$	$2w_4$	w_3	w_2	w_1
$y_2 \to -y_5$	w_7	w_5	$2w_3$	w_2	0	0	0	0	0	0	0	0
$y_4 to - y_6$	w_6	w_4	w_2	$2w_1$	0	0	0	0	0	0	0	0
$x_5 \to x_2$	0	0	0	0	0	u_4	0	u_3	0	u_2	u_1	0
$x_6 \to x_4$	0	0	0	0	0	0	u_4	0	u_3	0	u_2	u_1

16. Dominant Weight: $\lambda = [2, 1, -1, -1]$

- Weyl module : $V[1, 2, 0]$ of dimension 60.
- $L(\lambda)$ is a 60−dimensional quotient of $V[1, 2, 0]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 6J_1 \oplus 6J_2 \oplus 14J_3$.

$$J_1 : v_0 \ (u_6)$$

$$J_1 : y_1v_0 \ (u_5)$$

$$J_1 : y_2y_6v_0 + 2y_4y_5v_0 \ (u_4)$$

$$J_1 : y_1y_2y_6v_0 + 2y_1y_4y_5v_0 \ (u_3)$$

$$J_1 : y_2y_4y_5y_6v_0 \ (u_2)$$

$$J_1 : 2y_1y_2^{(2)}y_6^{(2)}v_0 + 2y_2y_4y_6^{(2)}v_0 \ (u_1)$$

$J_2 : y_2v_0 \to 2y_5v_0 \ (w_6)$

$J_2 : y_1y_2v_0 \to 2y_1y_5v_0 \ (w_5)$

$J_2 : y_4v_0 \to 2y_6v_0 \ (w_4)$

$J_2 : y_1y_4v_0 \to 2y_1y_6v_0 \ (w_3)$

$J_2 : 2y_1y_2^{(2)}y_6v_0 + 2y_2y_4y_6v_0 \to y_1y_2y_5y_6v_0 + y_2y_6^{(2)}v_0 + y_4y_5y_6v_0 \ (w_2)$

$J_2 : y_2^{(2)}y_4y_6^{(2)}v_0 \to 2y_2^{(2)}y_6^{(3)}v_0 \ (w_1)$

Action Map

	u_6	u_5	u_4	u_3	u_2	u_1	w_6	w_5	w_4	w_3	w_2	w_1
y_1	u_5	0	u_3	0	$2u_1$	0	w_5	$2w_3$	w_3	0	0	0
x_1	0	u_6	0	u_4	0	$2u_2$	0	$2w_6$	w_6	w_5	0	0
$h_{14} - h_{15}$	$2u_6$	0	0	u_3	u_2	$2u_1$	$2w_6$	0	0	w_3	w_2	$2w_1$
$h_{13} - h_{14} + h_{15}$	$2u_6$	$2u_5$	u_4	u_3	0	0	0	0	0	0	$2w_2$	w_1
$y_2 \to -y_5$	w_6	$2w_4 + w_5$	0	0	0	$2w_1$	0	u_4	0	u_3	u_2	0
$y_4 \to -y_6$	w_4	w_3	0	0	$2w_1$	0	$2u_4$	$2u_3$	0	0	$2u_1$	0
$x_5 \to x_2$	0	0	$w_4 + 2w_5$	$2w_3$	$2w_2$	0	u_6	u_5	u_5	0	0	$2u_1$
$x_6 \to x_4$	0	0	w_6	w_4	0	w_2	0	u_6	0	u_5	0	$2u_2$

17. Dominant Weight: $\lambda = [2, 1, -1, -2]$

- Weyl module : $V[1, 2, 1]$ of dimension 175.
- $L(\lambda)$ is a 156−dimensional quotient of $V[1, 2, 1]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 7J_1 \oplus 4J_2 \oplus 47J_3$.

$J_1 : y_2y_3v_0 + y_5v_0 \ (u_7)$

$J_1 : y_1y_2y_3v_0 + y_1y_5v_0 \ (u_6)$

$J_1 : y_1y_3y_4v_0 + 2y_1y_6v_0 \ (u_5)$

$J_1 : y_1y_2y_5y_6v_0 + y_1y_2^{(2)}y_3y_6v_0 + y_1y_4y_5^{(2)}v_0 + y_2y_3y_4y_6v_0 + 2y_2y_6^{(2)}v_0 + 2y_3y_4^{(2)}y_5v_0 \ (u_4)$

$J_1 : y_2y_4y_5^{(2)}y_6v_0 \ (u_3)$

$J_1 : 2y_1y_2^{(2)}y_5y_6^{(2)}v_0 + 2y_2y_4y_5y_6^{(2)}v_0 + y_2^{(2)}y_6^{(3)}v_0 \ (u_2)$

$J_1 : 2y_1y_2^{(2)}y_6^{(3)}v_0 + 2y_2y_4y_6^{(3)}v_0 \ (u_1)$

$J_2 : 1v_0 \to y_3v_0 \ (w_4)$

$J_2 : y_1v_0 \to y_1y_3v_0 \ (w_3)$

$J_2 : y_2^{(2)}y_4y_5y_6^{(2)}v_0 \to 2y_2^{(2)}y_5y_6^{(3)}v_0 \ (w_2)$

$J_2 : y_2^{(2)}y_4y_6^{(3)}v_0 \to 2y_2^{(2)}y_6^{(4)}v_0 \ (w_1)$

Action Map

	u_7	u_6	u_5	u_4	u_3	u_2	u_1	w_4	w_3	w_2	w_1
y_1	u_6	$2u_5$	0	0	$2u_2$	$2u_1$	0	w_3	0	w_1	0
x_1	0	$2u_7$	u_6	0	0	u_3	u_2	0	w_4	0	w_2
$h_{14} - h_{15}$	u_7	$2u_6$	0	0	0	u_2	$2u_1$	w_4	$2w_3$	w_2	$2w_1$
$h_{13} - h_{14} + h_{15}$	u_7	u_6	u_5	0	$2u_3$	$2u_2$	$2u_1$	0	0	0	0
$y_2 \to -y_5$	0	0	0	0	0	$2w_2$	$2w_1$	$2u_7$	u_6	0	0
$y_4 \to -y_6$	0	0	0	0	$2w_2$	w_1	0	u_6	$2u_5$	0	0
$x_5 \to x_2$	w_4	w_3	0	0	0	0	0	0	0	u_2	$2u_1$
$x_6 \to x_4$	0	w_4	w_3	0	0	0	0	0	0	$2u_3$	$2u_2$

18. Dominant Weight: $\lambda = [2, 1, -1, -3]$

- Weyl module : $V[1, 2, 2]$ of dimension 360.
- $L(\lambda)$ is a 360−dimensional quotient of $V[1, 2, 2]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 6J_1 \oplus 6J_2 \oplus 1147J_3$.

$J_1 : y_2y_4y_5^{(3)}y_6v_0\ (u_6)$

$J_1 : 2y_1y_2^{(2)}y_5^{(2)}y_6^{(2)}v_0 + 2y_2y_4y_5^{(2)}y_6^{(2)}v_0 + y_2^{(2)}y_5y_6^{(3)}v_0\ (u_5)$

$J_1 : y_2y_4y_5^{(2)}y_6^{(2)}v_0 + y_2^{(2)}y_3y_4y_5y_6^{(2)}v_0 + 2y_2^{(2)}y_5y_6^{(3)}v_0\ (u_4)$

$J_1 : 2y_1y_2^{(2)}y_5y_6^{(3)}v_0 + y_2y_4y_5y_6^{(3)}v_0 + y_2^{(2)}y_3y_4y_6^{(3)}v_0 + y_2^{(2)}y_6^{(4)}v_0\ (u_3)$

$J_1 : 2y_2y_4y_5y_6^{(3)}v_0 + y_2^{(2)}y_3y_4y_6^{(3)}v_0 + 2y_2^{(2)}y_6^{(4)}v_0\ (u_2)$

$J_1 : 2y_1y_2^{(2)}y_6^{(4)}v_0 + 2y_2y_4y_6^{(4)}v_0\ (u_1)$

$J_2 : 2y_1y_2^{(2)}y_5^{(2)}y_6v_0 + 2y_2y_4y_5^{(2)}y_6v_0 \to y_1y_2y_5^{(3)}y_6v_0 + 2y_2y_3y_4y_5^{(2)}y_6v_0\ (w_6)$

$J_2 : y_1y_2^{(2)}y_5y_6^{(2)}v_0 + 2y_2y_3y_4^{(2)}y_5y_6v_0 + 2y_2y_4y_5y_6^{(2)}v_0 \to$
$y_1y_2y_5^{(2)}y_6^{(2)}v_0 + y_1y_2^{(2)}y_3y_5y_6^{(2)}v_0 + 2y_2y_3y_4y_5y_6^{(2)}v_0 + y_2^{(2)}y_3^{(2)}y_4y_6^{(2)}v_0 + 2y_4y_5^{(2)}y_6^{(2)}v_0\ (w_5)$

$J_2 : y_1y_2^{(2)}y_6^{(3)}v_0 + 2y_2y_3y_4^{(2)}y_6^{(2)}v_0 \to 2y_1y_2y_5y_6^{(3)}v_0 + y_1y_2^{(2)}y_3y_6^{(3)}v_0 + y_2y_6^{(4)}v_0 + 2y_4y_5y_6^{(3)}v_0\ (w_4)$

$J_2 : y_2^{(2)}y_4y_5^{(2)}y_6^{(2)}v_0 \to 2y_2^{(2)}y_5^{(2)}y_6^{(3)}v_0\ (w_3)$

$J_2 : y_2^{(2)}y_4y_5y_6^{(3)}v_0 \to 2y_2^{(2)}y_5y_6^{(4)}v_0\ (w_2)$

$J_2 : y_2^{(2)}y_4y_6^{(4)}v_0 \to 2y_2^{(2)}y_6^{(}5)v_0\ (w_1)$

Action map

	u_6	u_5	u_4	u_3	u_2	u_1	w_6	w_5	w_4	w_3	w_2	w_1
y_1	$2u_5$	$u_3 + 2u_2$	u_3	u_1	u_1	0	$2w_5$	w_4	0	w_2	$2w_1$	0
x_1	0	0	u_6	$u_4 + u_5$	u_4	u_2	0	w_6	w_5	0	$2w_3$	w_2
$h_{14} - h_{13}$	$2u_6$	0	0	u_3	u_2	$2u_1$	$2w_6$	0	w_4	0	w_2	$2w_1$
$h_{13} - h_{14} + h_{15}$	u_6	u_5	u_4	u_3	u_2	u_1	0	0	0	$2w_3$	$2w_2$	$2w_1$
$y_2 \to -y_5$	0	$2w_3$	$2w_3$	0	w_2	$2w_1$	$2u_6$	u_5	$2u_3$	0	0	0
$y_4 \to -y_6$	$2w_3$	w_2	$2w_2$	w_1	0	0	$2u_5$	u_2	$2u_1$	0	0	0
$x_5 \to x_2$	$2w_6$	$2w_5$	0	w_4	w_4	0	0	0	0	$2u_5$	u_3	$2u_1$
$x_6 \to x_4$	0	w_6	0	$2w_5$	0	0	0	0	0	$2u_6$	u_4	u_2

19. Dominant Weight: $\lambda = [2, 0, 0, 0]$

- Weyl module : $V[2, 0, 0]$ of dimension 10.
- $L(\lambda)$ is a 10−dimensional quotient of $V[2, 0, 0]$.

Decomposition of $L(\lambda)$ under action of e_{43} : $3J_1 \oplus 2J_2 \oplus J_3$.

$$J_1 : v_0 \ (u_3)$$
$$J_1 : y_1 v_0 \ (u_2)$$
$$J_1 : y_1^{(2)} v_0 \ (u_1)$$
$$J_2 : y_4 v_0 \to y_6 v_0 \ (w_2)$$
$$J_2 : y_1 y_4 v_0 \to y_1 y_6 v_0 \ (w_1)$$

Action Map

	u_3	u_2	u_1	w_2	w_1
y_1	u_2	$2u_1$	0	w_1	0
x_1	0	$21v_0$	u_2	0	w_2
$h_{14} - h_{15}$	0	u_2	$2u_1$	w_2	$2w_1$
$h_{13} - h_{14} + h_{15}$	$2u_3$	$2u_2$	$2u_1$	0	0
$y_2 \to -y_5$	0	$2w_2$	$2w_1$	0	0
$y_4 \to -y_6$	w_2	w_1	0	0	0
$x_5 \to x_2$	0	0	0	u_2	$2u_1$
$x_6 \to x_4$	0	0	0	u_3	$2u_2$

20. Dominant Weight: $\lambda = [2, 0, 0, -1]$

- Weyl module : $V[2, 0, 1]$ of dimension 36.
- $L(\lambda)$ is a 36−dimensional quotient of $V[2, 0, 1]$.

Decomposition of $L(\lambda)$ under action of e_{43} : $6J_1 \oplus 6J_2 \oplus 6J_3$.

$$J_1 : y_5 v_0 \ (u_6)$$
$$J_1 : y_1 y_5 v_0 \ (u_5)$$
$$J_1 : y_3 y_4 v_0 + 2y_6 v_0 \ (u_4)$$
$$J_1 : y_1 y_3 y_4 v_0 + 2y_1 y_6 v_0 \ (u_3)$$
$$J_1 : y_1^{(2)} y_5 v_0 \ (u_2)$$
$$J_1 : y_1^{(2)} y_6 v_0 \ (u_1)$$
$$J_2 : 1v_0 \to y_3 v_0 \ (w_6)$$
$$J_2 : y_1 v_0 \to y_1 y_3 v_0 \ (w_5)$$
$$J_2 : y_1^{(2)} v_0 \to y_1^{(2)} y_3 v_0 \ (w_4)$$
$$J_2 : y_4 y_5 v_0 \to 2y_5 y_6 v_0 \ (w_3)$$
$$J_2 : y_1 y_4 y_5 v_0 \to 2y_1 y_5 y_6 v_0 \ (w_2)$$
$$J_2 : y_1 y_4 y_6 v_0 \to 2y_1 y_6^{(2)} v_0 \ (w_1)$$

Action Map

	u_6	u_5	u_4	u_3	u_2	u_1	w_6	w_5	w_4	w_3	w_2	w_1
y_1	u_5	$2u_2$	u_3	u_1	0	0	w_5	$2w_4$	0	w_2	$2w_1$	0
x_1	0	0	$2u_6$	u_5+u_4	$2u_5$	u_2	0	$2w_6$	w_5	0	$2w_3$	w_2
$h_{14}-h_{15}$	$2u_6$	0	0	u_3	u_2	$2u_1$	$2w_6$	0	w_4	0	w_2	$2w_1$
$h_{13}-h_{14}+h_{15}$	u_6	u_5	u_4	u_3	u_2	u_1	0	0	0	$2w_3$	$2w_2$	$2w_1$
$y_2\to -y_5$	0	$2w_3$	w_3	w_2	$2w_2$	$2w_1$	u_6	u_5+u_4	u_3+u_2	0	0	0
$y_4\to -y_6$	w_3	w_2	0	0	w_1	0	$2u_4$	$2u_3$	u_1	0	0	0
$x_5\to x_2$	w_6	w_5	$2w_5$	w_4	w_4	0	0	0	0	u_5+u_4	u_3+2u_2	$2u_1$
$x_6\to x_4$	0	w_6	w_6	0	w_5	w_4	0	0	0	u_6	$2u_5+u_4$	u_3

21. Dominant Weight: $\lambda = [2,0,0,-2]$

- Weyl module : $V[2,0,2]$ of dimension 84.
- $L(\lambda)$ is a 69−dimensional quotient of $V[2,0,2]$.

Decomposition of $L(\lambda)$ under action of e_{43} : $4J_1 \oplus 4J_2 \oplus 19J_3$.

$$J_1 : y_5^{(2)}v_0$$
$$J_1 : y_1y_5^{(2)}v_0$$
$$J_1 : y_1y_3y_4y_6v_0 + 2y_1y_6^{(2)}v_0$$
$$J_1 : y_1^{(2)}y_6^{(2)}v_0$$
$$J_2 : y_5v_0 \to y_3y_5v_0$$
$$J_2 : y_1^{(2)}y_6v_0 \to y_1^{(2)}y_3y_6v_0$$
$$J_2 : y_4y_5^{(2)}v_0 \to 2y_5^{(2)}y_6v_0$$
$$J_2 : y_1y_4y_6^{(2)}v_0 \to 2y_1y_6^{(3)}v_0$$

Action Map

	u_4	u_3	u_2	u_1	w_4	w_3	w_2	w_1
y_1	u_3	0	u_1	0	0	0	0	0
x_1	0	u_4	0	u_2	0	0	0	0
$h_{14}-h_{15}$	u_4	$2u_3$	u_2	$2u_1$	w_4	w_3	$2w_2$	$2w_1$
$h_{13}-h_{14}+h_{15}$	0	0	0	0	$2w_4$	$2w_3$	w_2	w_1
$y_2\to -y_5$	0	$2w_2$	0	$2w_1$	$2u_4$	$2u_2$	0	0
$y_4\to -y_6$	w_2	0	w_1	0	$2u_3$	$2u_1$	0	0
$x_5\to x_2$	w_4	0	w_3	0	0	0	$2u_3$	$2u_1$
$x_6\to x_4$	0	w_4	0	w_3	0	0	u_4	u_2

22. Dominant Weight: $\lambda = [2,0,-1,-1]$

- Weyl module : $V[2,1,0]$ of dimension 45.
- $L(\lambda)$ is a 45−dimensional quotient of $V[2,1,0]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 6J_1 \oplus 6J_2 \oplus 9J_3$.

$$J_1 : v_0 \ (u_6)$$
$$J_1 : y_1 v_0 \ (u_5)$$
$$J_1 : y_1^{(2)} v_0 \ (u_4)$$
$$J_1 : y_2 y_6 v_0 + 2y_4 y_5 v_0 \ (u_3)$$
$$J_1 : y_1 y_2 y_6 v_0 + 2y_1 y_4 y_5 v_0 \ (u_2)$$
$$J_1 : 2y_1 y_4 y_6 v_0 + 2y_1^{(2)} y_2 y_6 v_0 \ (u_1)$$
$$J_2 : y_2 v_0 \to 2y_5 v_0 \ (w_6)$$
$$J_2 : y_1 y_2 v_0 \to 2y_1 y_5 v_0 \ (w_5)$$
$$J_2 : y_4 v_0 \to 2y_6 v_0 \ (w_4)$$
$$J_2 : y_1 y_4 v_0 \to 2y_1 y_6 v_0 \ (w_3)$$
$$J_2 : y_1^{(2)} y_2 v_0 \to 2y_1^{(2)} y_5 v_0 \ (w_2)$$
$$J_2 : y_1^{(2)} y_4 v_0 \to 2y_1^{(2)} y_6 v_0 \ (w_1)$$

Action Map

	u_6	u_5	u_4	u_3	u_2	u_1	w_6	w_5	w_4	w_3	w_2	w_1
y_1	u_5	$2u_4$	0	u_2	$2u_1$	0	w_5	$2w_2$	w_3	$2w_1$	0	0
x_1	0	$2u_6$	u_5	0	$2u_3$	u_2	0	0	w_6	$w_4 + w_5$	$2w_5$	w_2
$h_{14} - h_{15}$	u_6	$2u_5$	0	$2u_3$	0	u_1	w_6	$2w_5$	$2w_4$	0	0	w_1
$h_{13} - h_{14} + h_{15}$	u_6	u_5	u_4	0	0	0	$2w_6$	$2w_5$	$2w_4$	$2w_3$	$2w_2$	$2w_1$
$y_2 \to -y_5$	w_6	$2w_4 + w_5$	$2w_3 + w_2$	0	0	0	0	u_3	u_3	u_2	u_2	u_1
$y_4 \to -y_6$	w_4	w_3	w_1	0	0	0	$2u_3$	$2u_2$	0	0	$2u_1$	0
$x_5 \to x_2$	0	0	0	$2w_4 + 2w_5$	$2w_3 + w_2$	$2w_1$	$2u_6$	$2u_5$	u_5	$2u_4$	$2u_4$	0
$x_6 \to x_4$	0	0	0	w_6	$2w_4$	$2w_3 + 2w_2$	0	$2u_6$	0	u_5	$2u_5$	$2u_4$

23. Dominant Weight: $\lambda = [2, 0, -1, -2]$

- Weyl module : $V[2, 1, 1]$ of dimension 140.
- $L(\lambda)$ is a 116−dimensional quotient of $V[2, 1, 1]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 4J_1 \oplus 8J_2 \oplus 32J_3$.

$$J_1 : y_3 y_4 v_0 + 2y_6 v_0 \ (u_4)$$
$$J_1 : y_1 y_3 y_4 v_0 + 2y_1 y_6 v_0 \ (u_3)$$
$$J_1 : y_2 y_5 y_6 v_0 + 2y_4 y_5^{(2)} v_0 \ (u_2)$$
$$J_1 : 2y_1 y_4 y_6^{(2)} v_0 + 2y_1^{(2)} y_2 y_6^{(2)} v_0 \ (u_1)$$
$$J_2 : v_0 \to y_3 v_0 \ (w_8)$$
$$J_2 : y_1 v_0 \to y_1 y_3 v_0 \ (w_7)$$
$$J_2 : y_1^{(2)} v_0 \to y_1^{(2)} y_3 v_0 \ (w_6)$$
$$J_2 : y_2 y_5 v_0 \to 2y_5^{(2)} v_0 \ (w_5)$$
$$J_2 : y_1 y_2 y_5 v_0 \to 2y_1 y_5^{(2)} v_0 \ (w_4)$$
$$J_2 : 2y_1 y_4 y_6 v_0 + 2y_1^{(2)} y_2 y_6 v_0 \to 2y_1 y_3 y_4 y_6 v_0 + y_1^{(2)} y_2 y_3 y_6 v_0 \ (w_3)$$

$J_2 : y_1^{(2)} y_4 y_6 v_0 \to 2y_1^{(2)} y_6^{(2)} v_0 \ (w_2)$

$J_2 : y_1 y_2 y_4 y_5 y_6 v_0 + 2y_2 y_3 y_4^{(2)} y_6 v_0 \to 2y_1 y_2 y_5 y_6^{(2)} v_0 + 2y_2 y_3 y_4 y_6^{(2)} v_0 \ (w_1)$

Action Map

	u_4	u_3	u_2	u_1	w_8	w_7	w_6	w_5	w_4	w_3	w_2	w_1
y_1	u_3	0	0	0	w_7	$2w_6$	0	w_4	0	w_2	0	0
x_1	0	u_4	0	0	0	$2w_8$	w_7	0	w_5	0	0	0
$h_{14} - h_{15}$	u_4	$2u_3$	u_2	u_1	0	w_7	$2w_6$	0	w_4	0	w_2	0
$h_{13} - h_{14} + h_{15}$	0	0	$2u_2$	$2u_1$	$2w_8$	$2w_7$	$2w_6$	w_5	w_4	w_3	w_2	0
$y_2 \to -y_5$	0	0	0	0	0	u_4	u_3	0	u_2	0	u_1	0
$y_4 \to -y_6$	0	0	0	0	$2u_4$	$2u_3$	0	$2u_2$	0	0	0	0
$x_5 \to x_2$	$2w_7$	w_6	$2w_4$	$2w_2$	0	0	0	0	0	0	0	0
$x_6 \to x_4$	$2w_8$	w_7	w_5	w_3	0	0	0	0	0	0	0	0

24. Dominant Weight: $\lambda = [2, 0, -1, -3]$

- Weyl module : $V[2, 1, 2]$ of dimension 300.
- $L(\lambda)$ is a 294−dimensional quotient of $V[2, 1, 2]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 8J_1 \oplus 8J_2 \oplus 90J_3$.

$J_1 : y_2 y_3 y_5 v_0 \ (u_8)$

$J_1 : y_1 y_2 y_3 y_5 v_0 \ (u_7)$

$J_1 : 2y_1 y_3 y_4 y_6 v_0 + y_1 y_6^{(2)} v_0 + y_1^{(2)} y_2 y_3 y_6 v_0 + y_1^{(2)} y_3 y_4 y_5 v_0 + y_1^{(2)} y_5 y_6 v_0 \ (u_6)$

$J_1 : y_1^{(2)} y_3 y_4 y_6 v_0 + 2y_1^{(2)} y_6^{(2)} v_0 \ (u_5)$

$J_1 : y_2 y_5^{(2)} y_6 v_0 + 2y_4 y_5^{(3)} v_0 \ (u_4)$

$J_1 : y_1 y_2 y_5^{(2)} y_6 v_0 + 2y_1 y_4 y_5^{(3)} v_0 \ (u_3)$

$J_1 : 2y_1 y_4 y_5 y_6^{(2)} v_0 + 2y_1^{(2)} y_2 y_5 y_6^{(2)} v_0 + y_3 y_4^{(2)} y_6^{(2)} v_0 \ (u_2)$

$J_1 : 2y_1 y_4 y_6^{(3)} v_0 + 2y_1^{(2)} y_2 y_6^{(3)} v_0 \ (u_1)$

$J_2 : y_2 y_5^{(2)} v_0 \to 2y_5^{(3)} v_0 \ (w_8)$

$J_2 : y_1 y_2 y_5^{(2)} v_0 \to 2y_1 y_5^{(3)} v_0 \ (w_7)$

$J_2 : 2y_2 y_3 y_4 y_5 v_0 + 2y_2 y_5 y_6 v_0 \to y_2 y_3 y_5 y_6 v_0 \ (w_6)$

$J_2 : 2y_1 y_2 y_3 y_4 y_5 v_0 + 2y_1 y_2 y_5 y_6 v_0 \to y_1 y_2 y_3 y_5 y_6 v_0 \ (w_5)$

$J_2 : y_1 y_2 y_3 y_4 y_6 v_0 + y_1 y_2 y_6^{(2)} v_0 + y_1^{(2)} y_2 y_5 y_6 v_0 + 2y_4 y_6^{(2)} v_0 \to$
$y_1 y_2 y_3 y_6^{(2)} v_0 + 2y_1 y_2 y_3^{(2)} y_4 y_6 v_0 + 2y_1 y_3 y_4 y_5 y_6 v_0 + 2y_1 y_5 y_6^{(2)} v_0 + y_1^{(2)} y_2 y_3 y_5 y_6 v_0 + y_1^{(2)} y_5^{(2)} y_6 v_0 + 2y_3 y_4 y_6^{(2)} v_0 \ (w_4)$

$J_2 : 2y_1 y_4 y_6^{(2)} v_0 + 2y_1^{(2)} y_2 y_6^{(2)} v_0 \to 2y_1 y_3 y_4 y_6^{(2)} v_0 + 2y_1^{(2)} y_2 y_3 y_6^{(2)} v_0 + y_1^{(2)} y_5 y_6^{(2)} v_0 \ (w_3)$

$J_2 : y_1^{(2)} y_4 y_5 y_6 v_0 \to 2y_1 y_3 y_4 y_6^{(2)} v_0 + 2y_1 y_6^{(3)} v_0 + 2y_1^{(2)} y_2 y_3 y_6^{(2)} v_0 \ (w_2)$

$J_2 : y_1^{(2)} y_4 y_6^{(2)} v_0 \to 2y_1^{(2)} y_6^{(3)} v_0 \ (w_1)$

	u_8	u_7	u_6	u_5	u_4	u_3	u_2	u_1	w_8	w_7	w_6	w_5	w_4	w_3	w_2	w_1
y_1	u_7	0	u_5	0	u_3	0	u_1	0	w_7	0	w_5	0	w_2+w_3	w_1	0	0
x_1	0	u_8	0	u_6	0	u_4	0	u_2	0	$2w_8$	$2w_8$	$2w_7$	0	$2w_4$	0	w_2+2w_3
$h_{14}-h_{15}$	$2u_8$	0	$2u_6$	0	0	u_3	0	u_1	$2w_8$	0	0	w_5	$2w_4$	0	0	w_1
$h_{13}-h_{14}+h_{15}$	$2u_8$	$2u_7$	$2u_6$	$2u_5$	u_4	u_3	u_2	u_1	0	0	0	0	0	0	0	0
$y_2\to -y_5$	w_8	w_7+w_8	w_4	w_2+w_3	0	0	0	0	0	u_4	$2u_4$	$2u_3$	0	u_2	$2u_2$	u_1
$y_4\to -y_6$	$2w_6$	$2w_5$	w_3	w_1	0	0	0	0	$2u_4$	$2u_3$	0	0	u_2	0	$2u_1$	0
$x_5\to x_2$	0	0	0	0	$2w_7+2w_8$	0	$2w_2+2w_3$	$2w_1$	u_8	u_7	0	0	$2u_6$	u_5	$2u_5$	0
$x_6\to x_4$	0	0	0	0	w_8	$2w_6$	$2w_4$	w_3	0	u_8	0	$2u_7$	0	0	u_6	u_5

25. Dominant Weight: $\lambda = [2, 0, -2, -2]$

- Weyl module : $V[2,2,0]$ of dimension 126.
- $L(\lambda)$ is a 126−dimensional quotient of $V[2,2,0]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 6J_1 \oplus 6J_2 \oplus 36J_3$.

$$J_1 : v_0 \ (u_6)$$
$$J_1 : y_1v_0 \ (u_5)$$
$$J_1 : y_1^{(2)}v_0 \ (u_4)$$
$$J_1 : y_2y_6v_0 + 2y_4y_5v_0 \ (u_3)$$
$$J_1 : y_1y_2y_6v_0 + 2y_1y_4y_5v_0 \ (u_2)$$
$$J_1 : y_1^{(2)}y_2y_6v_0 + 2y_1^{(2)}y_4y_5v_0 \ (u_1)$$
$$J_2 : y_2v_0 \to 2y_5v_0 \ (w_6)$$
$$J_2 : y_1y_2v_0 \to 2y_1y_5v_0 \ (w_5)$$
$$J_2 : y_4v_0 \to 2y_6v_0 \ (w_4)$$
$$J_2 : y_1y_4v_0 \to 2y_1y_6v_0 \ (w_3)$$
$$J_2 : y_1^{(2)}y_2v_0 \to 2y_1^{(2)}y_5v_0 \ (w_2)$$
$$J_2 : y_1^{(2)}y_4v_0 \to 2y_1^{(2)}y_6v_0 \ (w_1)$$

Action Map

	u_6	u_5	u_4	u_3	u_2	u_1	w_6	w_5	w_4	w_3	w_2	w_1
y_1	u_5	$2u_4$	0	u_2	$2u_1$	0	w_5	$2w_2$	w_3	$2w_1$	0	0
x_1	0	$2u_6$	u_5	0	$2u_3$	u_2	0	0	w_6	w_4+w_5	$2w_5$	w_2
$h_{14}-h_{15}$	$2u_6$	0	u_4	0	u_2	$2u_1$	$2w_6$	0	0	w_3	w_2	$2w_1$
$h_{13}-h_{14}+h_{15}$	0	0	0	$2u_3$	$2u_2$	$2u_1$	w_6	w_5	w_4	w_3	w_2	w_1
$y_2\to -y_5$	w_6	$2w_4+w_5$	w_2+w_3	0	0	0	0	u_3	u_3	u_2	u_2	u_1
$y_4\to -y_6$	w_4	w_3	w_1	0	0	0	$2u_3$	$2u_2$	0	0	$2u_1$	0
$x_5\to x_2$	0	0	0	w_4+2w_5	w_2+w_3	w_1	u_6	u_5	u_5	$2u_4$	u_4	0
$x_6\to x_4$	0	0	0	$2w_6$	w_4+w_5	w_3	0	u_6	$2u_6$	0	u_5	u_4

26. Dominant Weight: $\lambda = [2, 0, -2, -3]$

- Weyl module : $V[2,2,1]$ of dimension 360.
- $L(\lambda)$ is a 360−dimensional quotient of $V[2,2,1]$.

Decomposition of $L(\lambda)$ under action of $e_{43} : 6J_1 \oplus 6J_2 \oplus 114J_3$.

$$J_1 : y_2y_3v_0 + y_5v_0 \ (u_6)$$
$$J_1 : y_1y_2y_3v_0 + y_1y_5v_0 \ (u_5)$$

$J_1 : y_3y_4v_0 + 2y_6v_0 \ (u_4)$

$J_1 : y_1y_3y_4v_0 + 2y_1y_6v_0 \ (u_3)$

$J_1 : y_1^{(2)}y_2y_3v_0 + y_1^{(2)}y_5v_0 \ (u_2)$

$J_1 : y_1^{(2)}y_3y_4v_0 + 2y_1^{(2)}y_6v_0 \ (u_1)$

$J_2 : v_0 \to 2y_3v_0 \ (w_6)$

$J_2 : y_1v_0 \to 2y_1y_3v_0 \ (w_5)$

$J_2 : y_1^{(2)}v_0 \to 2y_1^{(2)}y_3v_0 \ (w_4)$

$J_2 : y_2y_3y_4v_0 + 2y_2y_6v_0 \to y_3y_4y_5v_0 + 2y_5y_6v_0 \ (w_3)$

$J_2 : y_1y_2y_3y_4v_0 + 2y_1y_2y_6v_0 \to y_1y_3y_4y_5v_0 + 2y_1y_5y_6v_0 \ (w_2)$

$J_2 : y_1^{(2)}y_2y_3y_4v_0 + 2y_1^{(2)}y_2y_6v_0 \to y_1^{(2)}y_3y_4y_5v_0 + 2y_1^{(2)}y_5y_6v_0 \ (w_1)$

Action Map

	u_6	u_5	u_4	u_3	u_2	u_1	w_6	w_5	w_4	w_3	w_2	w_1
y_1	u_5	$2u_2$	u_3	$2u_1$	0	0	w_5	$2w_4$	0	w_2	$2w_1$	0
x_1	0	0	u_6	u_4+u_5	$2u_5$	u_2	0	$2w_6$	w_5	0	$2w_3$	w_2
$h_{14}-h_{15}$	u_6	$2u_5$	$2u_4$	0	0	u_1	w_6	$2w_5$	0	$2w_3$	0	$2w_1$
$h_{13}-h_{14}+h_{15}$	$2u_6$	$2u_5$	$2u_4$	$2u_3$	$2u_2$	$2u_1$	w_6	w_5	w_4	0	0	0
$y_2 \to -y_5$	0	$2w_3$	w_3	w_2	$2w_2$	w_1	$2u_6$	$2u_5+u_4$	u_3+2u_2	0	0	0
$y_4 \to -y_6$	w_3	w_2	0	0	w_1	0	$2u_4$	$2u_3$	$2u_1$	0	0	0
$x_5 \to x_2$	w_6	w_5	$2w_5$	w_4	w_4	0	0	0	0	$2u_4+2u_5$	u_2+2u_3	$2u_1$
$x_6 \to x_4$	0	w_6	0	$2w_5$	w_5	w_4	0	0	0	u_6	$2u_4$	$2u_2+2u_3$

27. Dominant Weight: $\lambda = [2, 0, -2, -4]$

- Weyl module : $V[2, 2, 2]$ of dimension 729.
- $L(\lambda)$ is a 729−dimensional quotient of $V[2, 2, 2]$.

Decomposition of $L(\lambda)$ under action of e_{43} : $243J_3$.
Semisimplification is the zero map as all J_3 vanish after semisimplification.

Arjun Agarwal Portland, OR 97229
Email address: arjunagarwal010@gmail.com

Arun S. Kannan Chicago, IL 60606
Email address: arun_kannan@alum.mit.edu